VOL.
2
1961

Plasma Physics—Accelerators—Thermonuclear Research

JOURNAL OF NUCLEAR ENERGY: PART C

EDITOR-IN-CHIEF
J. V. DUNWORTH

GENERAL EDITORS
J. GUÉRON G. RANDERS

EXECUTIVE EDITORS
J. D. LAWSON P. REYNOLDS (U.K.)
H. HURWITZ JR. R. F. POST (U.S.A.)

EDITORIAL ADVISORY BOARD

J. B. ADAMS—Geneva
H. ALFVÉN—Stockholm
J. E. ALLEN—Rome
K. W. ALLEN—Aldermaston
T. E. ALLIBONE—Aldermaston
C. M. VAN ATTA—Livermore
P. R. BELL—Oak Ridge
R. J. BICKERTON—Harwell
J. P. BLEWETT—Brookhaven
S. C. BROWN—M.I.T.
Sir JOHN COCKCROFT—Cambridge
S. A. COLGATE—Livermore
E. D. COURANT—Brookhaven
J. D. CRAGGS—Liverpool
K. G. EMELEUS—Belfast
A. VON ENGEL—Oxford
R. A. FITCH—Aldermaston
B. H. FLOWERS—Manchester
D. W. FRY—Winfrith
I. N. GOLOVIN—Moscow
M. B. GOTTLIEB—Princeton
O. KOFOED-HANSEN—Röskilde
M. G. N. HINE—Geneva
Sir CHRISTOPHER HINTON—London
P. HUBERT—Fontenay-sur-Roses
D. L. JUDD—Berkeley
A. R. KANTROWITZ—Everett, Mass.
A. C. KOLB—Washington
R. LATHAM—London
M. A. LEONTOVICH—Moscow
M. STANLEY LIVINGSTON—Cambridge, Mass.
R. S. LIVINGSTON—Oak Ridge
G. MIYAMOTO—Tokyo
L. B. MULLETT—Harwell
R. S. PEASE—Harwell
T. G. PICKAVANCE—Harwell
D. H. ROBINSON—Burlington, Mass.
A. SCHLÜTER—Munich
B. F. J. SCHONLAND—Harwell
A. SIMON—Oak Ridge
L. SMITH—Berkeley
A. H. SNELL—Oak Ridge
L. SPITZER JR.—Princeton
K. R. SYMON—Madison
G. P. THOMSON—Cambridge
W. WALKINSHAW—Harwell
A. A. WARE—Aldermaston
C. N. WATSON-MUNRO—Sydney, N.S.W.
R. R. WILSON—New York
S. D. WINTER—Gif-sur-Yvette

VOLUME 2
1961

Proceedings of the

1959 INTERNATIONAL PLASMA PHYSICS INSTITUTE

PERGAMON PRESS
LONDON · OXFORD · NEW YORK · PARIS · LOS ANGELES

VOL
2
196

PROCEEDINGS OF THE

1959 INTERNATIONAL PLASMA PHYSICS INSTITUTE

SPONSORED BY

BOEING AIRPLANE COMPANY, NATIONAL SCIENCE FOUNDATION

UNIVERSITY OF WASHINGTON

VOL.
2
1961

Editor

JAMES E. DRUMMOND
Boeing Scientific Research Laboratories

Associate Editors

H. DREICER
Los Alamos Scientific Laboratory

P. A. STURROCK
Stanford University

L. WILETS
University of Washington

SYMPOSIUM PUBLICATIONS DIVISION

PERGAMON PRESS

OXFORD · LONDON · NEW YORK · PARIS

1961

VOL
2
196

LIST OF CONTENTS

Editors' introduction 1

I. QUANTUM PLASMA PHYSICS

D. PINES: Classical and quantum plasmas 5

J. J. QUINN and R. A. FERRELL: Quasi-particle approach to interaction in an idealized metal 18

D. TER HAAR: The Tomonaga method and some of its applications 25

H. E. DE WITT: Thermodynamic functions of a partially degenerate, fully ionized gas . 27

R. BROUT: Phonons and plasmons 46

A. E. GLASSGOLD: Collective oscillations of a system of interacting fermions . . . 51

C. J. POWELL: The origin of the characteristic electron energy losses in aluminium and magnesium 57

II. CLASSICAL PLASMA PHYSICS

K. G. EMELEUS: Plasma studies survey of experiments—I. Mainly optical studies . . 65

K. G. EMELEUS: Plasma studies survey of experiments—II. Mainly probe studies . . 69

J. R. PIERCE: Critical points in the theory of electron beam devices 73

T. COOR: Plasma diffusion in Stellarators 81

T. H. STIX: Ion cyclotron heating of a plasma 84

G. D. BOYD and R. W. GOULD: Travelling wave interaction in plasmas 88

W. E. DRUMMOND: Cyclotron radiation from a hot plasma 90

D. B. BEARD: Optical properties and emission of relativistic plasmas at cyclotron resonance 94

J. E. DRUMMOND, R. A. GERWIN and B. G. SPRINGER: The concept of conductivity . . 98

J. G. LINHART: Experiments on interaction of plasmas and microwaves 109

E. S. WEIBEL and G. L. CLARK: Confinement of an electron beam by oscillating electromagnetic fields 112

K. G. EMELEUS and D. W. MAHAFFEY: Plasma oscillations 117

O. BUNEMAN: Maintenance of equilibrium by instabilities 119

J. R. PIERCE: Instabilities and growing waves in electron beam devices 135

E. G. HARRIS: Plasma instabilities associated with anisotropic velocity distributions . . 138

E. N. PARKER: A quasi-linear model of plasma shock structure in a longitudinal magnetic field 146

C. OBERMAN: On the radiation from co-operative phenomena in plasmas 154

P. A. STURROCK: Non-linear effects in electron plasmas 158

S. J. BUCHSBAUM, E. I. GORDON and S. C. BROWN: Experimental study of a plasma column in a microwave cavity 164

VOL. 2 1961

III. STATISTICAL AND THERMODYNAMICAL PLASMA PHYSICS

R. Balescu: The approach to equilibrium of a plasma 169

E. P. Gross: Dynamics of classical many-body systems 173

I. Prigogine: On the motion of a charged particle 184

S. R. de Groot: Non-equilibrium thermodynamics of systems in an electromagnetic field . 188

N. Rostoker and M. N. Rosenbluth: Fokker-Planck equation for a plasma with a constant magnetic field 195

H. B. Levine: Classification of diagrams for a plasma equation-of-state 206

R. E. Kidder and H. E. de Witt: Application of a modified Debye-Hückel theory to fully ionized gases 218

G. M. Harris and J. Trulio: Equilibrium properties of a partially ionized plasma . . . 224

IV. SUMMARY TALK OF IDEAS PRESENTED AT THE 1959 INTERNATIONAL PLASMA PHYSICS INSTITUTE

M. N. Rosenbluth 235

Indexes 239

VOL
2
196

AUTHOR INDEX TO VOLUME 2

Balescu, R. p. 169
Beard, D. B. p. 94
Boyd, G. D. p. 88
Brout, R. p. 46
Brown, S. C. p. 164
Buchsbaum, S. J. p. 164
Buneman, O. p. 119

Clark, G. L. p. 112
Coor, T. p. 81

Drummond, J. E. p. 98
Drummond, W. E. p. 90

Emeleus, K. G. pp. 65, 69, 117

Ferrell, R. A. p. 18

Gerwin, R. A. p. 98
Glassgold, A. E. p. 51
Gordon, E. I. p. 164
Gould, R. W. p. 88
Groot, S. R. de p. 188
Gross, E. P. p. 173

Haar, D. ter p. 25
Harris, E. G. p. 138
Harris, G. M. p. 224

Kidder, R. E. p. 218

Levine, H. B. p. 206
Linhart, J. G. p. 109

Mahaffey, D. W. p. 117

Oberman, C. p. 154

Parker, E. N. p. 146
Pierce, J. R. pp. 73, 135
Pines, D. p. 5
Powell, C. J. p. 57
Prigogine, I. p. 184

Quinn, J. J. p. 18

Rosenbluth, M. N. pp. 195, 235
Rostoker, N. p. 195

Springer, B. G. p. 98
Stix, T. H. p. 84
Sturrock, P. A. p. 158

Trulio, J. p. 224

Weibel, E. S. p. 112
Witt, H. E. de pp. 27, 218

VOL.
2
1961

VOL
2
196

J Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 1 to 4. Pergamon Press Ltd. Printed in Northern Ireland

EDITORS' INTRODUCTION

ON August 31—September 5, 1959, an international conference on basic plasma physics, microwave plasma physics and the general many-body problem as related to plasmas was held at the University of Washington in Seattle. The sponsors of this Institute were the Boeing Airplane Company, the National Science Foundation and the University of Washington. Its purpose, as established by the committees which consisted of H. Dreicer, J. Drummond, W. Elsasser, R. Geballe, R. Gould, D. Kerst, E. Meeron, M. Rosenbluth and L. Wilets, was to provide reviews of recent developments in these related specialties. For this reason, all except four of the formal talks were one-half hour long with ten minutes additional for discussions. The time was monitored by a clock which started Tchaikovsky's 1812 Overture (with cannon) in a gradual crescendo after two warning lights had flashed on the podium. This was effective for most of the speakers, but one, Buneman, proved a match for all of the devices: he arranged his talk so that he could show a series of surrealistic type slides of electron trajectories to the accompaniment of the overture—a pleasant combination indeed.

VOL. 2 1961

The Institute was formally opened with a short welcoming address by Joseph McCarthy, Dean of the Graduate School, University of Washington. He admonished us to be mindful of the disciplines needed in physical science; a very appropriate reminder in view of the current impetus being given plasma studies by the thermonuclear goal. This attitude prevailed throughout most of the conference as thermonuclear 'machinery' was decidedly de-emphasized by the committee which extended the speaking invitations as well as by the speakers themselves.

The sessions on basic plasma physics were started with an introduction by D. Pines (General Atomic and University of Illinois) who presented a simple unified survey of collective and individual particle behaviour for both classical and quantum plasmas. The theory was illustrated by application to a fully-ionized, high-temperature electron-ion plasma, the motion of electrons and holes in semiconductors, the degenerate free electron gas and the effective interaction between electrons and ions in metals. This was followed by talks on the various theoretical approaches and methods.

M. Cohen (University of Chicago and General Electric) spoke on the relation of many-particle and one-particle treatments of collective phenomena. J. Quinn (R.C.A.) spoke on the quasi-particle approach to correlation in a degenerate electron gas. A quasi-particle is an electron or hole and its associated polarization cloud. As such it seems to represent an intermediate type of excitation between the extremes of a single electron and hole excitation and a plasma quantum or plasmon. Dr. Quinn used the quasi-particle approach in discussing effective mass, spin susceptibility and compressibility together with separation and ground state energies. D. ter Haar (Oxford) gave a brief account in classical language of Tomonaga's early and elegant but little-used method of introducing collective co-ordinates. He applied it to a derivation of the dispersion relation for plasma oscillations and to a derivation of the hydrodynamic equations of motion.

H. deWitt (U.C.R.L.) discussed a new derivation of the pressure and internal energy of a fully-ionized gas which are obtained directly in terms of density rather than chemical potential. The high-temperature classical limit equation of state is modified at lower temperatures by three types of corrections: direct exchange interactions, modification of the Debye-Hückel terms due to Fermi and Bose statistics and diffraction corrections due to the wave nature of the particles. R. Brout (Cornell) spoke on the interaction of phonons and plasmons in the random phase approximation. For low values of momentum transfer (compared to Fermi momentum) between the ion and electron vibrations phonons are found to decay into electron-hole pairs; but for large momentum transfers this does not occur. This effect might lead to determination of the Fermi surface as pointed out by W. Kohn and B. Knight. A. Glas gold (University of California) discussed the extension of plasma oscillation theory to the general problems of the collective excitations of a system of interacting fermions. For an infinite system of nucleons (nuclear matter) with two internal degrees of freedom (spin and isotopic spin), the collective oscillations are compressional waves (called 'zero-sound' by Landau), spin waves, isotopic spin waves and coupled spin-isotopic spin waves. When these waves are excited in a large nucleus by a supersonic particle, a characteristic radiation of nucleons and nuclear fragments is expected. In

addition to these results, which were obtained in collaboration with W. Heckrotte and K. M. Watson, there was also consideration given to Landau's theory of the 'Fermi liquid.'

A report of some excellent modern experimental work bearing on the origin of characteristic energy losses in aluminium and magnesium was given by C. J. Powell of the University of Western Australia. He showed that some of the inconsistencies in previous results could have been due to surface contamination of the specimens. His studies showed that the spectra of both materials were similar in that they were composed entirely of combinations of two elementary energy losses. The larger energy loss in each element was identified with the plasma loss proposed by Bohm and Pines, while the lower energy loss was identified with that proposed recently by Ritchie. The low lying losses were definitely indicated (by measurements made with very thin evaporated targets of aluminum) to be influenced (e.g. in position relative to the higher loss) by surface layers of the specimen.

The sessions on Tuesday were begun by reviews of experimental optical and probe techniques by K. G. Emeleus (The Queen's University, Belfast, Northern Ireland). He (very appropriately in these days of sophisticated and expensive equipment) admonished us to look at the electromagnetic radiation coming from our electrical discharges—with our eyes even! Thus we might avoid some of our grosser misconstructions of reality. He also dealt with the significance of gas purity and the molecular gas problem; the quantal limitation imposed on the time resolution by the Einstein coefficients for allowed and forbidden transitions; formulation and exploitation of the electron excitation integral; and further examples of spectroscopic technique, including use of interference filters, a check on the voltage amplitude of plasma electron oscillations and a very interesting and useful non-perturbing technique for studying potentials in sheaths. He made some general remarks about Langmuir probes and their construction; the Langmuir-Druyvesteyn theory for determining near-isotropic distributions, the problem of variable reflection coefficients and of Boyd's recent theoretical work; the problem of the study of oscillating plasmas; effects of negative ions on plasma fields and probe currents and on the stability of discharges.

J. R. Pierce (Bell Telephone Laboratories) discussed critical points in the theory of electron beam devices. Some of these critical problems also occur in plasma theory and we benefitted by learning both the approaches, insights and solutions as well as the occasional failures of the microwave electronics physicists. He discussed such matters as growing waves by the two-stream instability and parametric amplification by a moving discontinuity.

T. Coor (Princeton) provided us with a description of the failure of plasma diffusion theory to account for observations on confinement times. He showed that the randomly varying electric fields (noise as measured by double probes) present in the stellarator plasmas is of sufficient intensity to give order of magnitude agreement between the plasma drifts produced by these fields and the observed drifts. He also put forward a new hypothesis to explain the origin of the observed high-amplitude noise which might be applicable to gas discharges generally. It is assumed that the non-Maxwellian electron distribution resulting from the applied electric field excites large-amplitude, coherent-plasma oscillations near the plasma frequency. These large-amplitude oscillations act as the 'pump' in the non-linear process known as parametric amplification. Thermal fluctuations, normally present with small amplitudes in quiescent plasmas, are amplified over a wide band of frequencies, both above and below the pump frequency. T. Stix (Princeton) discussed experiments showing qualitative agreement with the theory of ion cyclotron heating of a plasma. R. Gould (California Institute of Technology) discussed the theory and some of his very interesting and provocative experiments on the interaction of electron beams and plasmas. J. Phillips (Los Alamos Scientific Laboratory) discussed recent experimental results on stabilized pinch discharges.

A very lively argument developed on Wednesday morning between the brothers Drummond, on confinement of radiation from very hot plasmas. W. E. Drummond (General Atomic) presented the results of a theoretical study made by himself and M. Rosenbluth which showed the validity of the assumption made by Trubnikov and Kudryavtsev in their Geneva paper on cyclotron (harmonic) radiation from a hot plasma. The Drummond and Rosenbluth results show that the individual electrons in a plasma radiate as though they were in a vacuum whenever the square of the radiation frequency is much greater than the square of the plasma frequency as it is for the most important components of the radiation. In addition Drummond and Rosenbluth estimated the spread of the emitted radiation, and finding it quite small calculated a very substantial reduction in the critical size of a thermonuclear reactor employing an efficient specular reflector. J. E. Drummond challenged the significance of this result by pointing out that if their assumption of perfect specularity were relaxed by only a small amount there would result mixing of the

VOL 2 196

radiation between various angles which would build up important emission intensities from all angles (as a consequence of the small but non-vanishing absorption of the plasma at all angles). This effect is expected to be small if only a few hundred reflections are needed. D. Beard (University of California) spoke next on the same topic. He had estimated the cyclotron harmonic resonance radiation from relativistic plasmas by computing the absorption of an incident plane wave and invoking Kirchhoff's relation. Next J. E. Drummond (Boeing) presented a generalization of conductivity tensor for a plasma. He, together with R. Gerwin and B. Springer, had shown that in general a plasma is transconductive and had derived from the linearized Boltzmann equation a fairly general formula for the high frequency conduction kernel for quasi-steady state inhomogeneous plasmas. In addition they had shown that within this model the necessary and sufficient condition for the total coherent power transfer between a plasma and the electromagnetic field to be zero is that the kernel be anti-hermitian. Examples for homogeneous and inhomogeneous plasmas were presented. A general resonance theory was also presented and a new electrodynamic equation combining Maxwell's equations with a solution form of the Boltzmann equation was derived. W. E Drummond arose to point out that there was still a lot of calculating necessary to apply this theory to given physical problems.

J. Linhart (CERN) described two experiments on the interaction of plasmas and microwaves. In the first, he showed the existence of three classes of modes of oscillation: a cavity-mode, a coaxial mode and a purely plasma mode. A comparison with a theory was made. In the second, a hollow plasma shell was produced at the periphery of a cylindrical resonator oscillating in one of its resonant modes. The shell was magnetically driven in constricting the region of oscillation, and the resulting increase of frequency was observed as for Einstein's pendulum. E. Weibel (Space Technology Laboratories) showed the experimental verification by G. L. Clark and himself of theoretical predictions on the confinement of an electron beam by oscillating electromagnetic fields. An electron beam which diverges strongly because of Coulomb repulsion is suddenly and quite dramatically confined to within 0·2 cm by a pulse of microwave energy.

The only parallel session was an informal and very lively one on the basic many-body problem, held on Wednesday morning. G. E. Uhlenbeck (Brookhaven) discussed in some detail the current problems in this field. R. Brout (Cornell) presented a beautifully simple theory of the random ferromagnet, using his semi-invariant technique and diagrammatic representation. The method permits, for example, a practically instantaneous derivation of the Mayer cluster expansions. N. M. Hugenholtz (Princeton) discussed field theoretic techniques as related to the many-body problem.

A very beautiful Wednesday afternoon was spent on a cruise on Puget Sound.

All Thursday and Friday morning were devoted to statistical physics. I. Prigogine (Universite Libre de Bruxelles), described a new and rigorous theory of irreversible processes based on a diagrammatic description of the Fourier expansion of the Liouville distribution function. Application of this theory to the derivation of a rigorous evolution equation for plasmas was beautifully presented by R. Balescu (Universite Libre de Bruxelles). The equation has a formal similarity with the usual Fokker–Planck equation, but Coulomb interaction between two particles is replaced by an effective interaction which depends on the distribution of all other particles. For plasmas we may validly employ the Balescu integral equation which, however, is quite difficult to solve.

In general, diagrammatic techniques seem to be in fashion nowadays. Problems of classification of diagrams for quantum statistics of plasmas were discussed by H. B. Levine (Lawrence Radiation Laboratory) and further development of the graph-theoretical technique of nodal expansions was reported by E. Meeron. While it appears that the nodal expansion method represents a further generalization of the random phase approximation, it is difficult to determine the physical meaning of the various terms involved.

A completely different approach, using the consistent field theory, was presented by E. P. Gross (Brandeis University) who succeeded in treating close collisions in his theory, thus generalizing the Landau damping mechanism to include the case of close collisions.

S. R. de Groot (University of Leyden) gave a review of the macroscopic theory of irreversible processes in fluid systems in an electromagnetic field.

I. Oppenheim (National Bureau of Standards) derived the equations for singlet and pair distribution functions by use of a superposition approximation for the triplet function. The singlet equation is a modified Boltzmann equation expressing the effects of mean long-range forces as well as short-range collisions. The mean forces are expanded in parameter related to the gradients of macroscopic properties of the system and the results should be valid for plasmas which are not too far from equilibrium.

VOL. 2 1961

N. Rostoker (General Atomic) presented the elegant and important new statistical theory by M. Rosenbluth and himself. They have derived from the Liouville equation a hierarchy of equations for S-body distribution functions near stable meta-equilibria, but unlike the Yvon-Born-Green hierarchy, each equation depends for its solution on a simpler rather than a more difficult equation. Thus they build up from a Boltzmann-Fokker-Planck type equation rather than down from the Liouville equation. They have found an expansion parameter which expresses the validity of any given equation in the hierarchy; it is the reciprocal of the number of electrons in a Debye sphere.

G. E. Uhlenbeck (Brookhaven) presented an outline of a new version, due to R. Guernsey and himself, of Bogolyubov's theory for completely-ionized gases.

R. Mjolsness (Los Alamos) discussed particle velocity distributions in a fully-ionized gas emphasizing the question of the production and relaxation of a doubly-peaked distribution, giving some numerical results for a plasma in externally applied electric and magnetic fields as worked out by H. Dreicer and himself.

M. Kac (Cornell) presented a study of a model of N interacting points in a finite cube. He has shown that the grand partition function is the fundamental solution of a certain parabolic partial differential equation in a space of n dimensions in the limit $n \to \infty$.

N. Reiss (Bell Laboratories) described some new techniques in the statistical mechanism of fluids and B. Alder (University of California, Berkeley) spoke very interestingly on condensed plasmas showing, for instance, that iodine should be expected to become a semiconductor under very high pressures.

Friday afternoon was spent discussing beams in plasmas. K. G. Emeleus spoke on plasma oscillations emphasizing the recent experimental work of D. W. Mahaffey and himself, pointing out the importance of quasi-static density gradients. O. Buneman (Stanford and Cambridge Universities) described the role that collective beam instabilities can play in establishing near-Maxwellian distributions from grossly non-Maxwellian ones within a few plasma periods. He presented numerical calculations for the development of the instabilities in the non-linear regime. J. R. Pierce (Princeton) provided us with some of the wealth of examples of instabilities and growing waves in electron beam devices. J. Dawson (Princeton) discussed plasma oscillations of a large number of electron beams. E. G. Harris (University of Tennessee) spoke on instabilities of transverse as well as longitudinal waves in plasmas with anisotropic velocity distributions.

The final sessions were on Saturday morning. E. N. Parker (University of Chicago) showed that the two-stream electrostatic instability should lead to the formation of shock waves in tenuous plasmas. The characteristic distance in the relative stream velocity times the ion plasma period. He also showed that the presence of a magnetic field introduces another coupling and a characteristic shock thickness of the ion Larmor radius. C. Oberman (Princeton) discussed some properties of the transmission, reflection and radiation of bounded plasmas in a strong magnetic field. J. Dawson (Princeton) described results of numerical calculations on the thermalization of large amplitude plasma oscillations. They show that for an initially cold plasma, the ordered motion was 60 per cent randomized within the first three oscillation periods. For an initially tepid plasma, 90 per cent randomization of the wave motion occurred in the first oscillation period. An interesting by-product of the calculation was the observation that a few particles gained energies in the order of 10 times the average energy. P. Sturrock (Stanford) discussed non-linear effects pointing out how the Hamiltonian formalism leads naturally to expressions for power transfer and frequency shifts in terms of 'collisions' between various normal wave modes in the plasma.

S. Buchsbaum (Bell Laboratories) discussed the experimental microwave generation of a column of plasma in a cavity, and the use of higher cavity modes to measure properties of the plasma.

A summary of some of the ideas presented at the Institute was given by M. Rosenbluth of General Atomic, and is included here following the other papers.

VOL
2
196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 5 to 17. Pergamon Press Ltd. Printed in Northern Ireland

I. QUANTUM PLASMA PHYSICS

CLASSICAL AND QUANTUM PLASMAS*

D. PINES†
John Jay Hopkins Laboratory for Pure and Applied Science, General Atomic Division of General Dynamics Corporation, San Diego, California

Abstract—A brief survey is given of the different theoretical approaches to the study of the low-density, high-temperature 'classical' plasma and of the high-density, low-temperature 'quantum' plasma. The random phase approximation, collective-variables method is shown to yield the same result as the collisionless Boltzmann equation combined with the self-consistent field method. Dynamic behaviour is studied in the classical plasma, with particular attention to the relationship between individual particle and collective behaviour. The frequency and wave number dependent longitudinal dielectric constant of the plasma is used to establish the many similarities in behaviour of quantum and classical plasmas in the random phase approximation; the validity of the random phase approximation is discussed. Applications of the theory are made to the ground state energy of the quantum plasma and to the explanation of the characteristic energy losses in solids. The generalizations required to take into account the ionic motion are indicated and the results are described.

VOL. 2 1961

1

THEORETICAL work on classical and quantum plasmas in the absence of a magnetic field has tended to fall into two major categories in recent years:

(i) Attempts to carry out rigorous derivations of plasma properties in the limit of high temperatures and low densities on the one hand and, on the other, of low temperatures and high densities (the high-density, free-electron gas model of a metal). Such attempts have received, on the quantum side especially, great impetus from the application of field-theoretic techniques to the many-body problem, and have been characterized by considerable success.

(ii) Efforts to reconcile and understand the simultaneous appearance of collective and individual particle-like behaviour in the plasma. Such efforts have their historical origin in the pioneering work of Debye on the screening action and Langmuir and Tonks on the oscillatory behaviour of classical plasmas. Again, it can be said that the recent work has been marked by success, in that it is now possible to outline a unified theory of collective and individual-particle behaviour for a wide variety of plasma phenomena.

In this talk I shall endeavour to indicate the present status of work along these lines: I shall try particularly to indicate the connexions between the different theoretical approaches.

I shall begin with a discussion of the classical electron plasma, that is, a gas of interacting electrons in a uniform back-ground of positive charge, and then go on to develop the quite close relationship which exists between the classical and quantum plasma. I shall also discuss briefly the generalization of the ideas developed to the 'plasma' of electrons moving in the periodic ionic potential of a solid. In the consideration of the classical plasma, I shall concentrate on the physical properties and dynamic behaviour of the system and shall not attempt to discuss the work on the equilibrium properties such as the free energy, specific heat and the like. For the quantum plasma, the physical properties, including the energy of the ground state, will be of particular concern. The considerations of the plasma of electrons in solids will be directed toward an understanding of the excitation spectrum of the system which is observed by, for instance, experiments on the scattering of fast electrons by thin solid films. Finally, the generalizations required to take into account the ionic motion are indicated for both the classical and quantum plasma, and the results are discussed.

2

There have been two basic, and at first sight quite different approaches developed for the treatment of classical plasmas: an individual-particles approach due to VLASOV (1945) and LANDAU (1946); and a collective-variable approach due to BOHM and PINES (1952). The individual-particles approach begins with the collisionless Boltzmann equation for the one-particle distribution function, $f(\mathbf{r}\mathbf{v}t)$:

$$\frac{\partial f}{\partial t}(\mathbf{r}\mathbf{v}t) + \mathbf{v} \cdot \nabla f(\mathbf{r}\mathbf{v}t) - \frac{eE}{m} \cdot \nabla_{\mathbf{v}} f(\mathbf{r}\mathbf{v}t) = 0. \quad (2.1)$$

* This work was done under a contract between General Atomic and the Texas Atomic Energy Research Foundation.

† Permanent address: Departments of Physics and Electrical Engineering, University of Illinois, Urbana, Illinois.

Here **E** is the average electric field arising from the interaction between the electrons, which is determined in a self-consistent way by using Poisson's equation,

$$\text{div } \mathbf{E} = 4\pi\rho = -4\pi e \int f(\mathbf{rv}t)\, d\mathbf{v}. \qquad (2.2)$$

These equations may then be solved simply if one regards the departure of f from its equilibrium Maxwellian value, f_0, as being slight; thus write $f = f_0 + \delta f$, and neglect the terms in (2.1) which are of order $\mathbf{E} \cdot \nabla_{\mathbf{v}} \delta f$. If one takes the Fourier transform in space and time of (2.1) and (2.2), one then arrives at a consistency condition which is the dispersion relation for the collective plasma oscillations,

$$1 + \frac{4\pi e^2}{mk^2} \int \frac{d\mathbf{v}\mathbf{k} \cdot \nabla_{\mathbf{v}} f_0(\mathbf{v})}{\omega - \mathbf{k} \cdot \mathbf{v} + i/\tau} = 0. \qquad (2.3)$$

Remark that in carrying out the Fourier-analysis, one does not automatically have a prescription for dealing with the poles at $\mathbf{k} \cdot \mathbf{v} = \omega$ in the above expression; the appropriate prescription was given first by LANDAU who treated the problem as an initial-value one, using Laplace transforms. The addition of a small imaginary part, i/τ, to the frequency is equivalent to LANDAU's prescription, and corresponds to the use of retarded solutions of the equations (2.1) and (2.2). The pole corresponds to the possibility of an individual particle moving at the phase velocity of the plasma wave exchanging energy with the wave and the foregoing prescription for a Maxwellian distribution implies that the collective modes will consequently be damped by the excitation of individual particle motion.

The dispersion relation, (2.3), may easily be solved in the limit of small k, that is $kv_- \ll \omega$, where v_- is defined by $mv_-^2 = KT$. One then finds, on writing $\omega = \omega_1 - i\omega_2$,

$$\omega_1^2 \cong \omega_p^2 + 3k^2 v_-^2 + \cdots \qquad (2.4)$$

$$\frac{\omega_2}{\omega_1} = \sqrt{\frac{\pi}{8}} \frac{\omega_1^3}{k^3 v_-^3} \exp\left(-\frac{\omega_1^2}{2k^2 v_-^2}\right) \cong \sqrt{\frac{\pi}{8}} \frac{k_d^3}{k^3} \exp\left(-\frac{k_d^2}{2k^2}\right) \qquad (2.5)$$

where the plasma frequency, ω_p, and the Debye screening wave-vector, k_d, are defined by:

$$\omega_p^2 = \frac{4\pi ne^2}{m}; \qquad k_d^2 = \frac{4\pi ne^2}{KT} = \frac{\omega_p^2}{v_-^2}.$$

Thus for $k \ll k_d$, one has nearly undamped plasma oscillations at a frequency of approximately ω_p. One finds for $k \cong k_d$, $\omega_1 \approx \omega_2$ and the oscillations become strongly damped by the individual electron excitations; for $k \gg k_d$, $\omega_2 \gg \omega_1$, so that no vestige of plasma oscillation remains. Thus the Debye wave vector serves as the maximum wave vector beyond which one can no longer regard the plasma oscillation as a well defined excitation of the system.

An approach which is closely related to the above is that of BOHM and GROSS (1949). Instead of working with the continuous distribution, $f(\mathbf{rv}t)$, they characterize the plasma as a set of beams of particles with discrete velocities $\mathbf{v}_i$. They proceed to solve in self-consistent fashion the linearized version of Poisson's equation and Liouville's equation. Their treatment differs from the foregoing in that they single out the particles moving with velocities $\mathbf{v}_i$ such that $\mathbf{k} \cdot \mathbf{v}_i = \omega$. These particles are then treated separately by going to the co-ordinate system travelling with the phase velocity of the plasma waves, in which the particles may be regarded as trapped in a static potential; the damping of the wave due to these trapped particles is then calculated. Thus BOHM and GROSS arrive at a dispersion relation which is essentially identical with (2.3); their separation of the particles into those which are not trapped and those which are trapped corresponds to writing

$$\int \frac{d\mathbf{v}F(\mathbf{v})}{\omega - \mathbf{k} \cdot \mathbf{v} + i/\tau} = \text{P.P.} \int \frac{d\mathbf{v}F(\mathbf{v})}{\omega - \mathbf{k} \cdot \mathbf{v}} - i\pi \int d\mathbf{v}F(\mathbf{v})\, \delta(\omega - \mathbf{k} \cdot \mathbf{v});$$

that is, to a distinct treatment of the principal part and the δ-like singularity in (2.3).

Let us turn now to a consideration of the theories which treat the collection motion first, by finding the appropriate collective variables of the plasma, and then take into account the individual-particle-like behaviour. The procedure of BOHM and PINES is the following. First, one can analyse the general physical behaviour of the system by studying the equations of motion of the density fluctuations, ρ_k, defined by

$$\rho_k = \int \rho(\mathbf{x}) \exp(-i\mathbf{k} \cdot \mathbf{x})\, d\mathbf{x} = \sum_i \exp(-i\mathbf{k} \cdot \mathbf{x}_i). \qquad (2.6)$$

These are

$$\ddot{\rho}_k + \omega_p^2 \rho_k = -\sum_i (\mathbf{k} \cdot \mathbf{v}_i)^2 \exp(-i\mathbf{k} \cdot \mathbf{x}_i) - \sum_{k' \neq k} \frac{4\pi e^2}{m} \frac{\mathbf{k} \cdot \mathbf{k}'}{(k')^2} \rho_{k'-k} \rho_{k'}. \qquad (2.7)$$

The equation (2.7) possesses a simple approximate solution if one makes the random phase approximation (RPA) of BOHM and PINES. In this approximation, the coupling between density fluctuations of different wave-vector is neglected. The RPA is equivalent to linearizing the equations of motion of the ρ_k; it acquired this name because in the second term

on the r.h.s. of (2.7) one has a product of density fluctuations, each of which may be expected to be small, since each $\rho_k' = \sum_i \exp(i\mathbf{k}' \cdot \mathbf{x}_i)$ is the sum of exponential factors with randomly varying phases.

Within the RPA, the ρ_k are nearly the correct oscillatory variables at long wavelengths ($k \ll k_d$), since in this limit the first term on the r.h.s. of (2.7) can clearly be neglected. This term, which is present even in the absence of interaction between the particles, describes the random thermal motion of the individual particles. It is obviously the dominant term at large wave-vectors ($k \gg k_d$), so that the simple analysis of the equations of motion shows again that k_d is the dividing line between collective and individual particle behaviour. Actually, the ratio, $k^2 v_-^2/\omega_p^2$, is a measure of the coupling of the kinetic motion of the electrons with their collective motion brought about by the Coulomb interaction. One can take into account the effect of this coupling within the RPA by going over to a proper set of collective variables; that is, a set of variables, $\xi_{k,\omega}$, which satisfy the equation of motion

$$\dot{\xi}_{k,\omega} + i\omega\xi_{k,\omega} = 0 \tag{2.8}$$

and so describe independent collective oscillation. The $\xi_{k,\omega}$'s are defined by

$$\xi_{k,\omega} = \sum_i \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_i} \exp(-i\mathbf{k} \cdot \mathbf{x}_i) \tag{2.9}$$

and obey the equation of motion (2.8) provided one makes the RPA, and provided ω satisfies the dispersion relation

$$1 = \frac{4\pi e^2}{m} \sum_c \frac{1}{(\omega - \mathbf{k} \cdot \mathbf{v}_i)^2}. \tag{2.10}$$

If one now replaces the summation in (2.10) by an average over the Maxwellian distribution at a temperature T, one obtains a dispersion relation which is almost that of (2.3); it differs from (2.3), in that it reproduces only the contribution from the principal part of the integral appearing there.

The reason for this discrepancy is very simple; one has not allowed for the possibility of damped waves. In fact, the $\xi_{k,\omega}$'s of (2.9) are not even well-defined, since they become infinite when $\mathbf{k} \cdot \mathbf{v}_i = \omega$. This difficulty may easily be remedied by adding a small imaginary part, i/τ, to the frequency ω. Thus one works instead with

$$\xi'_{k,\omega} = \sum_i \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_i + i/\tau} \exp(-i\mathbf{k} \cdot \mathbf{x}_i) \tag{2.9'}$$

and obtains a dispersion relation,

$$1 = \frac{4\pi e^2}{m} \sum_i \frac{1}{(\omega - \mathbf{k} \cdot \mathbf{v}_i + i/\tau)^2}, \tag{2.10'}$$

which upon taking a Maxwellian average is identical with the dispersion relation, (2.3).

Once one has found the appropriate collective variables, $\xi_{k,\omega}$, one can determine the individual-particle aspect of the plasma. This may be done by carrying through a consistent separation of the density fluctuation, ρ_k, into two parts,

$$\rho_k = q_k + \eta_k, \tag{2.11}$$

where q_k represents the collective component associated with the plasma oscillation and η_k the individual-particle component. The collective component is determined up to a normalization constant by the fact that

$$q_k \sim \frac{\xi_{k,\omega} - \xi_{k,-\omega}}{2}$$

satisfies an oscillatory equation of motion,*

$$\ddot{q}_k + \omega^2 q_k = 0. \tag{2.12}$$

The normalization constant is determined by requiring that the separation be consistent, that is, that the η_k display a typical individual-particle-like equation of motion, with no collective aspect present within the RPA. One then finds

$$q_k = \sum_i \frac{\omega_p^2}{\omega^2 - (\mathbf{k} \cdot \mathbf{v}_i)^2} \exp(-i\mathbf{k} \cdot \mathbf{x}_i) \tag{2.13}$$

$$\eta_k = \sum_i \frac{\omega^2 - \omega_p^2 - (\mathbf{k} \cdot \mathbf{v}_i)^2}{\omega^2 - (\mathbf{k} \cdot \mathbf{v}_i)^2} \exp(-i\mathbf{k} \cdot \mathbf{x}_i) \tag{2.14}$$

The η_k may be shown to represent a collection of individual electrons surrounded by co-moving clouds of charge which screen the field of the electrons within a distance of the order ot the Debye length.

Thus once one has taken into account the collective behavior characterized by the plasma oscillations, one finds automatically the screening aspect of the electron gas. What is left-over once the plasma oscillations are described explicitly is a collection of individual electrons necessarily surrounded by a screening cloud in such a way that their effective interaction with one another, or with an external charge, is screened within a distance of the order of the Debye length. It should be remarked that the foregoing separation of the density fluctuation into the components is essentially only useful for $k \lesssim k_d$, since only for this range of k do the collective modes represent a nearly independent mode of excitation of the system. For $k \gtrsim k_d$, the

* We shall not extend the generalizations, (2.9′) and (2.10′) which involve $[\omega + i/\tau]$ in place of ω, to the considerations of the remainder of this section, or to the one following. The generalizations are obvious; for the physical applications we make here, they are not of importance.

plasma oscillations are strongly damped, and the density fluctuations are those corresponding to a collection of individual, weakly-interacting electrons.

Another collective variables method which is closely related to the foregoing involves the explicit introduction of collective variables into a Hamiltonian formulation of the problem. This may be done by adding a set of terms involving potential collective co-ordinates to the basic Hamiltonian (BOHM, 1959); it is also likely that it can be accomplished by a judicious choice of canonical transformations, along the lines of the recent work of LIPKIN (1959). In either case, after a series of canonical transformations one arrives at a Hamiltonian involving both collective variables and individual electron co-ordinates. The collective part of the Hamiltonian describes a set of independent plasma waves at frequency ω, while the individual electron part describes the interaction of a collection of electrons plus their associated screening clouds. The physical content of this approach is thus identical to the foregoing. It possesses certain advantages, in that a somewhat more systematic investigation of corrections to the RPA is possible within the Hamiltonian framework. It also possesses certain complications in dealing with the excited states of the system, because one has too many independent variables. One must therefore impose a set of subsidiary conditions to take this aspect of the problem into account; I shall not here enter upon a discussion of the role which these play.*

3

It is instructive to analyse a specific problem from the two points of view: the individual particle and the collective variables approach. Let us consider the 'classic' problem of a particle of charge Ze moving through the plasma at a constant velocity, $\mathbf{V}_0$. As the particle moves through the plasma, it will be surrounded by a polarization cloud which acts to screen its interaction with the individual electrons. The induced polarization cloud acts on the particle itself; this action gives rise to an alteration of the self-energy of the particle, and a slowing down of the particle due to the electric field of the polarization cloud.

The interaction of the charged particle with the plasma may be described by a potential energy term

$$\sum_i -\frac{4\pi Ze^2}{k^2}\rho_k \exp i\mathbf{k}\,.\,(\mathbf{R}_0 - \mathbf{V}_0 t) \qquad (3.1)$$

where $\mathbf{R}_0$ is the initial position of the particle, and we have made the approximation of constant velocity $\mathbf{V}_0$ to characterize the charged particle orbit in the plasma. In the collective variables approach, one treats separately the regions $k \leq k_d$ and $k \geq k_d$. For $k \leq k_d$, one computes the collective response to the particle explicitly, according to:

$$\ddot{q}_k + \omega^2 q_k = -\omega_p^2 \exp i\mathbf{k}\,.\,(\mathbf{R}_0 - \mathbf{V}_0 t) \qquad (3.2)$$

If the particle moves with a velocity $\mathbf{V}_0$ such that $k_d V_0 \leq \omega$, one finds the collective, plasma wave, response is such as to screen out the field of the particle within a distance of the order of the Debye length and alter its self-energy in a way which is equivalent to an increase in the particle mass. The increase corresponds to the effective mass of the screening cloud of plasma waves which the particle carries along with it. In fact, as is shown in Section 2, the individual-particles component, η_k, of the density fluctuation just corresponds to the electrons plus their associated plasma wave screening clouds. On the other hand, for $V_0 > \omega/k_d$, the particle will excite plasma oscillations for those wave-vectors k such that $\mathbf{k}\,.\,\mathbf{V}_0 = \omega$. The induced polarization density then takes the form of a wake behind the particle, analogous to the Cerenkov wake observed for a charged particle moving through a dielectric with a velocity greater than the velocity of light in the medium. The plasma wave contribution to the stopping power of the plasma for the charged particle is given by

$$\left(\frac{dT}{dx}\right)_{\text{coll.}} = \frac{4\pi Ze^4}{mV_0^2} \ln \frac{k_d V_0}{\omega_p}. \qquad (3.3)$$

For $k \leqslant k_d$, the individual-particle response to the charged particle will be small, because the η_k, which describe it, correspond to electrons whose interaction with the charged particle is strongly screened by their associated polarization clouds. On the other hand, for $k \geqslant k_d$, the individual-particle response will be the only response of importance; it may be calculated to a reasonable degree of approximation by neglecting all screening behaviour in the plasma. The individual-particle contribution to the stopping power is then:

$$\left(\frac{dT}{dx}\right)_{\text{i.p.}} \cong \frac{4\pi Ze^4}{mV_0^2} \ln \frac{k_{\max}}{k_d} \qquad (3.4)$$

where $k_{\max}$ is the appropriate cut-off wave-vector, given by the inverse of the distance of closest approach, or by the Broglie wave-vector of the charged particle, whichever enters first.

Results identical to the foregoing may be obtained from the Hamiltonian formulation of the problem.

A compact way to characterize the individual particles approach to the charged particle problem is

* For a detailed discussion of the role of the subsidiary conditions, see BOHM (1959).

by the introduction of the longitudinal frequency and wave-number dependent dielectric constant of the plasma, $e(\mathbf{k}, \Omega)$. The usefulness of such an approach has been particularly emphasized by LINDHARD (1954); as we shall see, the dielectric constant also provides a simple way to compare the classical and the quantum plasma, and plays an especially useful role in the consideration of the ground state of the quantum plasma. Consider the Fourier transform in space and time of Poisson's equations for an external charge in the plasma:

$$-k^2\epsilon(k, \Omega)\phi(k, \Omega) = 4\pi\overset{\text{ext.}}{\rho}(k, \Omega) \tag{3.5}$$

$$-k^2\phi(k, \Omega) = 4\pi\left[\overset{\text{ext.}}{\rho}(k, \Omega) - e\langle\rho(k, \Omega)\rangle\right]. \tag{3.6}$$

Here $\phi(k, \Omega)$ is the electric field potential, $\overset{\text{ext.}}{\rho}(k, \Omega)$ is the Fourier transform of the external charge density, and $\langle\rho(k, \Omega)\rangle$ the transform of the average induced density fluctuation in the plasma. $\epsilon(k, \Omega)$ characterizes the system response to an external charge, and one has

$$\frac{1}{\epsilon(k, \Omega)} = 1 + \frac{\langle -e\rho(k, \Omega)\rangle}{\overset{\text{ext.}}{\rho}(k, \Omega)}. \tag{3.7}$$

For the particle of charge Ze moving at velocity $\mathbf{V}_0$,

$$\overset{\text{ext.}}{\rho}(k, \Omega) = Ze\,\delta(\Omega - \mathbf{k}\cdot\mathbf{V}_0), \tag{3.8}$$

according to (3.1). Using (3.5) and (3.8) one may express the induced polarization density, $\langle\rho(x)\rangle$, the charged particle self-energy in the plasma, ΔE, and the stopping power of the plasma, dW/dx, in terms of $\epsilon(k, \mathbf{k}\cdot\mathbf{V}_0)$. One finds:

$$\langle\rho(x) = -Z\sum_k \left\{\frac{1}{\epsilon(k, \mathbf{k}\cdot\mathbf{V}_0)} - 1\right\} \exp i\mathbf{k}\cdot(\mathbf{r} - \mathbf{R}_0\mathbf{V}_0 t) \tag{3.9}$$

$$\Delta E = -\sum_k \frac{2\pi Z^2e^2}{k^2}\left\{\frac{1}{\epsilon(k, \mathbf{k}\cdot\mathbf{V}_0)} - 1\right\} \tag{3.10}$$

$$\frac{dT}{dx} = -\sum_k \frac{4\pi Z^2e^2}{k^2V_0}\,\mathbf{k}\cdot\mathbf{V}_0\,\mathrm{Im}\,\frac{1}{\epsilon(k, \mathbf{k}\cdot\mathbf{V}_0)} \tag{3.11}$$

where the symmetry properties of $\epsilon(k, \Omega)$ are invoked to obtain the stopping power expression, (3.11), involving only the imaginary part of $\epsilon^{-1}(k, \mathbf{k}\cdot\mathbf{V}_0)$.

We may further remark that the zero's of $\epsilon(k, \Omega)$ yield the dispersion relation for the plasma oscillations,

$$\epsilon(k, \omega) = 0. \tag{3.12}$$

To see this, consider Poisson's equations, (3.5) and (3.6). In the absence of an external charge, one has two kinds of solutions:

(a) $\epsilon(k, \Omega) \neq 0$, $\phi(k, \Omega) = 0$, $\langle\rho(k, \Omega)\rangle = 0$, corresponding to zero average density fluctuation or potential fluctuation in the absence of an external charge.

(b) $\epsilon(k, \Omega) = 0$, with non-zero values of $\phi(k, \Omega)$, $\rho(k, \Omega)$, corresponding to the presence of non-zero density and potential fluctuations associated with the plasma oscillations, even in the absence of an external charge. One may calculate $\epsilon(k, \Omega)$ using the collisionless Boltzmann equation, as Lindhard has done. The result is

$$\underset{\text{RPA}}{\epsilon(k, \Omega)} = 1 + \frac{4\pi e^2}{mk^2}\int d^3\mathbf{v}\,\frac{\mathbf{k}\cdot\nabla_{\mathbf{v}}f_0(\mathbf{v})}{\Omega - \mathbf{k}\cdot\mathbf{v} + i/\tau}, \tag{3.13}$$

as might have been anticipated from the foregoing remarks combined with (2.3). This result possesses a suggestive physical interpretation in terms of an extension of the Lorentz calculations of the dielectric constant and polarizability (LORENTZ, 1952). It corresponds to

$$\underset{\text{RPA}}{\epsilon(k, \Omega)} = 1 + 4\pi\alpha(k, \Omega) \tag{3.14}$$

where $\alpha(k, \Omega)$ is the classical free electron polarizability; this is the result one would anticipate in the absence of potential corrections to $\alpha(k, \Omega)$ and of local field corrections. I have used the subscript RPA in (3.14) because this value may be derived directly within the RPA, by computing the response of the density fluctuations response to a test charge, and then carrying out the appropriate statistical averages.* Indeed, the use of the dielectric constant affords a simple way to connect the individual particles approach with the RPA.

A convenient expression for evaluating $\epsilon(k, \Omega)$ is the following:

$$\epsilon(k, \Omega) = 1 + \frac{k_d^2}{k^2}\,W\left(\frac{\omega}{kv_-}\right) \tag{3.15}$$

where

$$W(x) = \lim_{\delta\to 0} -\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\,\frac{y\exp(-y^2/2)}{x - y + i\delta}$$

$$= 1 + i\sqrt{\frac{\pi}{2}}\,x\exp(-x^2/2) - x\exp(x^2/2)\int_0^x dy\exp y^2/2. \tag{3.16}$$

* The calculation is carried out by generalizing the quantum mechanical method of NOZIÈRES and PINES (1958a) to finite temperatures and then going to the classical limit. *See* F. ENGLERT *J. Phys. Chem. Solids.* To be published.

With the aid of (3.15) and (3.16) one may easily obtain the following low-frequency and high-frequency expansions for $\epsilon(k, \Omega)$:

$$(\Omega \ll kv_-) \quad \epsilon = 1 + \frac{k_d^2}{k^2}\left(1 - \frac{\Omega^2}{k^2 v_-^2} + \ldots + i\sqrt{\frac{\pi}{2}}\frac{\Omega}{kv_-}\exp - \frac{\Omega^2}{2k^2v_-^2}\right) \tag{3.17}$$

$$(\Omega \gg kv_-) \quad \epsilon = 1 - \frac{\omega_p^2}{\Omega^2} \times\left(1 + \frac{3k^2v_-^2}{\Omega^2} + \frac{15k^4v_-^4}{\Omega^4} + \ldots\right) + i\sqrt{\frac{\pi}{2}}\frac{k_d^2}{k^2}\frac{\Omega}{kv_-}\exp - \frac{\Omega^2}{2k^2v_-^2} \tag{3.18}$$

The foregoing expressions for $\epsilon(k, \Omega)$ permit the straight-forward evaluation of the response of the plasma to the charged particle. If the charged particle moves at a speed $V_0 \ll v_-$, the screening cloud may be computed directly from (3.9) and (3.17), and corresponds simply to the Debye-Hückel cloud. In this same limit the self-energy, ΔE, may be written

$$\Delta E = -A_0 + A_1\frac{P_0^2}{2M} + \ldots$$

where P_0 is the momentum of the charged particle, M its mass and A_0 and A_1 are constants. Evaluation of A_1 shows that the screening cloud gives rise to an effective increase in the particle mass which is:

$$\frac{\delta M}{M} = \frac{mZ^2}{3M}\frac{e^2k_D}{KT}. \tag{3.19}$$

The result (3.19) is of the same order of magnitude as that calculated from (3.2) considering only the collective response to particle motion.

In the individual particles approach the distinction between the collective aspect and the individual-particle aspect arises in the following way. Consider the stopping power of the plasma, which may be written from (3.11) as:

$$\frac{dT}{dx} = -\sum_k \frac{4\pi Z^2e^2}{k^2V_0}\mathbf{k}\cdot\mathbf{V}_0\frac{\epsilon_2(k, \mathbf{k}\cdot\mathbf{V}_0)}{\epsilon_1^2(k, \mathbf{k}\cdot\mathbf{V}_0) + \epsilon_2^2(k, \mathbf{k}\cdot\mathbf{V}_0)} \tag{3.20}$$

where ϵ_1 and ϵ_2 are the real and imaginary parts of the RPA dielectric constant, (3.13). There are then two distinct contributions to the stopping power for a fast charged particle. These may be simply calculated in the limit $V_0 \gg v_-$. The first contribution arises from that region in k-space for which $\epsilon_1(k, \mathbf{k}\cdot\mathbf{v}_0) = 0$ while at the same time $\epsilon_2(k, \mathbf{k}\cdot\mathbf{V}_0) \ll 1$. This is a typical resonance contribution and is to be associated with the emission of plasma waves of frequency $\omega = \mathbf{k}\cdot\mathbf{V}_0$; since the condition $\epsilon(k, \omega) = 0$ is the dispersion relation for plasma oscillations. $\epsilon_1(k, \mathbf{k}\cdot\mathbf{V}_0) = 0, \epsilon_2(k, \mathbf{k}\cdot\mathbf{V}_0) \ll 1$ is just the condition that the waves be weakly damped and is, according to (3.18) and (3.25), reasonably well-satisfied in the region $k \lesssim k_d$. The plasma wave contribution to the stopping power under these circumstances may then be shown to be identical with the value (3.3), obtained using the collective variables approach.

The individual electron contribution to the stopping power arises from the non-resonance regions of k-space for which $\epsilon_2(k, \mathbf{k}\cdot\mathbf{V}_0)$ is appreciable. Inspection then shows that for $k \lesssim k_d$, the individual particles contribution is strongly screened; thus $\epsilon_1 \sim k^2/k_d^2$, $\epsilon_2 \sim k^2/k_d^2$, so that the individual particles contribution in this domain may be neglected in first approximation. On the other hand, for $k > k_d$, the screening is much reduced, so that one obtains a reasonable answer by approximating the denominator in (3.20) by unity. The result so obtained is then seen to be identical with that expected for a gas of non-interacting electrons, and agrees in turn with (3.4).

4

The two basic approaches we have been considering are in fact characterized by a number of approximations. In the self-consistent field-collisionless Boltzmann equation method, there are two approximations which require justification:

(1) The use of the collisionless Boltzmann equation combined with the self-consistent field method to obtain the one-particle distribution function.

(2) The linearization of the resulting system of coupled equations, a procedure which assumes that the departure of the distribution function from its equilibrium value is small.

In the collective-variables approach, again there are two approximations made:

(1) The coupling between different Fourier components of the Coulomb interaction is neglected in computing the influence of electron interaction on the plasma properties.

(2) The equilibrium distribution function is used to carry out an evaluation of the dispersion relation for collective modes, an approximation which is equivalent to assuming the actual departure of the system properties from their non-interacting values is slight.

There is, in a sense, a one-to-one correspondence between the approximations (1) for the two methods, and again for the approximations (2). The principal

VOL 2 196

difference between the methods would appear to lie in the fact that in the individual-particles approach one begins with a statistical description of the system, and then makes the approximations described, while in the collective-variables approach one makes the major approximation directly on the equations of motion for the quantities of interest, and subsequently carries out a statistical averaging process.

We shall, rather loosely, refer to the collected set of approximations as the random phase approximation, principally because the use of the term, 'collisionless Boltzmann equation-self-consistent field-random phase linear approximation, might be more accurate, but is somewhat unwieldly; it is by now clear that the different approaches yield the same results.

What can we now say about the validity of the RPA? Recently, ROSTOKER and ROSENBLUTH (to be published) have derived the basic equations of the VLASOV–LANDAU approach from the Liouville equation in 6–N dimensional space, by carrying out a systematic expansion procedure in which only terms of lowest order in e^2k_D/KT are retained. From their work, then, we can conclude that the RPA calculations of plasma properties will be valid when

$$\frac{e^2k_D}{KT} \ll 1.$$

The expansion parameter, e^2k_D/KT, is an eminently reasonable one, being the ratio of the effective average particle potential energy to the average particle kinetic energy; the smallness of e^2k_D/KT would appear a necessary condition for the validity of the RPA assumption that the Coulomb interactions, when properly taken into account, have only a slight influence on the equilibrium statistical properties of the system. We remark further that

$$\frac{e^2k_D}{KT} \sim (k_d r_0)^3 \sim \frac{n'}{n};$$

the success of the RPA requires that the Debye length be large compared to the average inter-particle spacing r_0, or that the number of independent collective plasma modes ($n' \sim k_d^3$) be small compared to the total number of particle degrees of freedom.

Detailed calculations of the corrections to the RPA description of the dynamic behaviour of the plasma have not yet been carried out. We may, however, obtain an idea of the nature of some of the higher-order corrections on the basis of existing calculations. Suppose we consider, as did LINDHARD (1954) the modification in the motion of a given electron as a consequence of its interaction with the other electrons in the plasma. An estimate of the extent of this modification may be obtained by applying (3.19) to the motion of a slow electron; we see that an increase in effective mass occurs, of order e^2k_D/KT. Thus in next-order the individual electron behaviour will be more nearly that of a gas of electrons of mass, $m^* = m[1 + \alpha(e^2k_D/KT)]$ where α is a constant.

This result is consistent with the estimate of BOHM and PINES (1952) of the change in plasma frequency produced by corrections to the RPA. They computed the influence of the neglected term on the r.h.s. of (2.7) on the frequency of oscillation of the ρ_k, and found that it gave rise to corrections of order

$$\frac{\delta\omega}{\omega_p} \sim \frac{k^2v_-^2}{\omega_p^2}\left(\frac{e^2k_D}{KT}\right). \tag{4.1}$$

As we have seen the coupling between the kinetic motion of the electrons and the potential-induced motion gives rise within the RPA to corrections to ω_p which are of order (k^2v^2/ω_p^2). The correction to the RPA thus appears in this term, and is of order e^2k_D/KT. This is just the correction one would obtain by carrying out the statistical average of (2.10) with a Maxwellian gas of free electrons of mass $m^* = m[1 + \alpha e^2k_D/KT]$.

It is worth emphasizing that for most classical plasmas of physical interest (ranging from the ionosphere to the hoped for fusion-reactors), the criterion $e^2k_D/KT \leqslant 1$ is well-satisfied, so that the RPA constitutes a good starting point for a description of the plasma properties. However a certain amount of caution should be exercized with respect to one aspect of plasma behaviour, the study of the build-up of growing plasma waves as a consequence of an instability in the plasma. The possible existence of growing waves may be studied successfully with the aid of the RPA dispersion relation, (2.3) or (3.12), suitably generalized to take into account departures of the electron velocity distribution from Maxwellian, the motion of the ions, a drift of electrons, etc. However, once one encounters a growing wave situation, there is no reason to expect that the smallness of e^2k_D/KT constitutes an adequate criterion for the validity of growth rates calculated from the RPA dispersion relation. Implicit in the use of the RPA is the assumption that one deals with a thermal level of plasma oscillation. As large amplitude plasma oscillations build up, non-linear effects due to interactions between plasma modes of different wave-vector (effects which lie outside the RPA) will almost certainly play an important role.

5

Let us now consider the quantum plasma, with particular attention to the ground-state properties of

the system. The quantum plasma is the well-known free electron gas model of a metal, in which the periodic array of positive ions is replaced by a uniform background of positive charge. One may also expect to meet with examples of quantum plasmas in suitable circumstances in both semi-conductors and semimetals. There has been a particularly large outpouring of theoretical papers on the high-density electron gas in recent years and I shall not attempt a detailed survey of recent theoretical work. Rather, I shall attempt to indicate the major lines of development, and then discuss the physical properties of the system, with particular attention to the close relationship which exists between the classical plasma and the quantum plasma.

Again, there have been two main lines of approach to the problem, corresponding to an individual particles approach and a collective variables approach. The individual particles approach has been developed in a variety of guises. Calculations of the dynamic behaviour of the system have been carried out using the collisionless quantum-mechanical Boltzmann equation combined with the self-consistent field method by KLIMONTOVITCH and SILIN (1952), by LINDHARD (1954) and by COHEN and EHRENREICH (to be published) amongst others. A direct calculation of the dielectric constant, $\epsilon(k, \Omega)$ by computing the response of the many-electron system to an oscillating test charge has been carried out by NOZIÈRES and PINES (1958a) within the RPA, which again is equivalent to treating the different wave-vectors, or momentum transfers, associated with the Coulomb interaction on a completely independent basis. The RPA calculation is in agreement with the self-consistent field calculations. NOZIÈRES and PINES have further shown that the ground state energy may be obtained directly from a knowledge of $\epsilon(k, \Omega)$ for all frequencies, wave-vectors and for all values of the coupling constant, e^2, in the problem. This affords a link between the above calculations and the applications of field-theoretic techniques to the calculations of the ground state energy which have been made by (HUBBARD (1957 and 1958), GELL-MANN and BRUECKNER (1957) and DU BOIS (1959) within the framework of the many-body perturbation theory. In this fashion one sees directly that the high-density limit calculations of these authors is the RPA.

Collective variables methods have been developed by BOHM and PINES (1953), SAWADA *et al.* (1957) and others. The BP calculations were carried out by direct analysis of the equations of motion, as in Section 3, and by a Hamiltonian formulation in which collective variables are explicitly introduced to describe the plasma oscillations. SAWADA *et al.* show how the equation of motion analysis combined with the introduction of collective variables may be extended to yield the ground-state energy of the electron gas within the RPA. As to the connexions between the different approaches, SAWADA *et al.* have shown that their version of the RPA is directly equivalent to GELL-MANN and BRUECKNER. NOZIÈRES and PINES (1958b), have shown explicitly that the BP Hamiltonian formulation also yields results in agreement with GELL-MANN and BRUECKNER and, finally, COHEN and EHRENREICH have demonstrated explicitly the connexion between the method of SAWADA and the quantum version of the method of the self-consistent field plus the collisionless Boltzmann equation.

The quantum dielectric constant within the RPA may be written as

$$\underset{\text{RPA}}{\epsilon(k, \Omega)} = 1 + 4\pi\alpha(k, \Omega) \tag{5.1}$$

where $\alpha(k,\Omega)$ is the free-electron polarizability, given by the Kraners-Heisenberg formula. For the ground state we have

$$\epsilon(k,\Omega) = 1 + \frac{8\pi e^2}{k^2}\left\{\sum_{\substack{K<k_0 \\ |\mathbf{K}+\mathbf{k}|>k_0}} \frac{\left(\frac{\hbar\mathbf{K}\cdot\mathbf{k}}{m} + \frac{\hbar k^2}{2m}\right)}{\left(\frac{\hbar\mathbf{K}\cdot\mathbf{k}}{m} + \frac{\hbar k^2}{2m}\right)^2 - \Omega^2} + i\pi \sum_{\substack{K<k_0 \\ |\mathbf{K}+\mathbf{k}|>k_0}} \delta\left(\frac{\hbar\mathbf{K}\cdot\mathbf{k}}{m} + \frac{\hbar k^2}{2m} - \Omega\right)\right\} \tag{5.2}$$

where the sums over K refer to all one-electron transitions which are consistent with the Pauli principle for a collection of electrons initially confined to a sphere of radius $\mathbf{k}_0$, the wave-vector at the Fermi surface. For symmetry reasons, the Pauli principle restriction, $|\mathbf{K} + \mathbf{k}| > k_0$ in the real part of $\epsilon(k, \Omega)$ can in fact be ignored. A somewhat more compact way of expressing $\epsilon(k, \Omega)$, which will be useful for later purposes, is

$$\epsilon(k, \Omega) = 1 + \frac{4\pi e^2}{m}\left\{\sum_n \frac{f_{0n}}{\omega_{n0}^2 - \Omega^2} + i\pi \sum_n \frac{f_{0n}}{\omega_{0n}}\,\delta(\omega_{n0} - \Omega)\right\} \tag{5.3}$$

where we have introduced the one electron excitation frequencies, ω_{n0}, and the oscillator strengths, f_{0n}, defined by

$$f_{0n} = \frac{2m}{\hbar k^2}\,\omega_{n0}(\rho_k)_{n0}^2.$$

The states 0 and n refer to the ground state of the free electron gas (all electrons inside the Fermi sphere) and

the one-electron excited states which may be reached from it by application of the operator ρ_k. An analytic expression for $\epsilon(k, \Omega)$ may be found in LINHARD (1954).

We first consider the dispersion relation for plasma oscillations, $\epsilon(k, \omega) = 0$. The condition that the real part of ϵ vanish may be put in the form derived first by BOHM and PINES,

$$1 = \frac{4\pi e^2}{m} \sum_i \frac{1}{(\omega - \mathbf{k} \cdot \mathbf{v}_i)^2 - \frac{\hbar^2 k^4}{4m^2}}, \qquad (5.4)$$

where the sum over i is over all particles lying within the unperturbed Fermi sphere. This dispersion relation differs from (2.10) only in the presence of the recoil term, $\hbar^2k^4/4m^2$; at long wavelengths, then, the plasma dispersion relation for the quantum plasma is just that of the classical plasma, before one carries out the statistical average over the unperturbed velocity distribution. The expansion of (5.4) in the long wavelength limit leads to the quantum dispersion relation,

$$\omega^2 \cong \omega_p^2 + \frac{3}{5} k^2 V_0^2 + \frac{\hbar^4 k^4}{4m^2} + \ldots \qquad (5.5)$$

where $\mathbf{V}_0$ is the velocity of an electron on the Fermi surface. Thus far we have neglected the possible damping of the plasma oscillations, arising from ϵ_2. The reason is that for long wavelengths there will be no damping; as is clear from (5.2), ϵ_2 remains zero until one reaches a wave-vector k such that $\hbar\mathbf{K} \cdot \mathbf{k}/m + \hbar k^2/2m = \omega$, that is, such that energy and momentum can be conserved for the decay of a plasmon (the quantum of plasma oscillation) into a single electron excitation which begins for an electron initially in the unperturbed Fermi distribution. This critical wave-vector k_c is obviously defined by

$$\frac{\hbar k_0 k_c}{m} + \frac{\hbar k_c^2}{2m} = \omega. \qquad (5.6)$$

The solution of (5.6) has been discussed by FERRELL (1957). For our purposes it suffices to replace ω by ω_p and to neglect the recoil term, $\hbar k_c^2/2m$. We then find

$$k_c \cong 0{\cdot}47\, r_s^{1/2}\, k_0, \qquad (5.7)$$

where r_s is the inter-electron spacing measured in units of the Bohr radius, defined according to

$$n = \frac{1}{(4\pi r_s^3 a_0^3/3)}.$$

For metallic electron densities, r_s varies between 1·8 and 5·5, so that the plasmon is stable (within the RPA) for wave-vectors which are less than or comparable to k_0.

The screening action of the quantum plasma is clear when one considers the low frequency ($\Omega \ll kV_0$) expansion of $\epsilon(k,\Omega)$:

$$\epsilon(k,\Omega) = 1 + \frac{k_{FT}^2}{k^2}\left(1 + \frac{i\pi}{2}\frac{\Omega}{kV_0} + \ldots\right), \qquad (5.8)$$

where k_{FT}^2 is the Fermi-Thomas screening wave-vector, the quantum analogue of the Debye wave-vector k_d. It is defined as

$$k_{FT}^2 = \frac{3\omega_p^2}{V_0^2} \quad \text{or} \quad k_{FT} = 0{\cdot}81\, r_s^{1/2}\, k_0. \qquad (5.9)$$

In the quantum plasma, then, one finds the effective interaction between the individual electrons is screened, with a screening radius of the order of k_{FT}^{-1}. Again, too, one finds that the screening wave-vector is of the same order of magnitude as the maximum wave-vector for moderately damped plasma oscillation. Again, the collective variables approach of BOHM and PINES (1953) shows that when the independent plasmon modes are separated out, there remains a part of the Hamiltonian which describes the effective interaction between the electrons plus their associated screening clouds.

One can proceed to determine the response of the quantum plasma to the passage of a fast charged particle in very much the same way as we have done for the classical case (LINDHARD, 1954; HUBBARD, 1955; NOZIÈRES and PINES, 1959). The results will be quite similar, because of the great similarity in the dielectric constants, in that one finds primarily plasmon excitation for momentum transfers up to $k \cong k_c$, and primarily individual electron excitation at the larger momentum transfers. One obvious difference is that the energy transfer to the plasma oscillations is now quantized, and takes place in multiples of the energy in a plasmon quantum $\hbar\omega_p$. In Section 6 we consider briefly the question of plasmon excitation in solids.

In order to obtain the ground-state energy from the dielectric constant, one makes use of the following identity (NOZIÈRES and PINES, 1958),

$$E_{\text{int.}} = \langle \Psi_0 | \sum_k \frac{2\pi e^2}{k^2} (\rho_k \rho_{-k} - n) | \Psi_0 \rangle$$
$$= \sum_k \left\{ \frac{\hbar}{2\pi} \int_0^\infty d\Omega\, \text{Im}\, \frac{1}{\epsilon(k, \Omega)} - \frac{2\pi n e^2}{k^2} \right\}. \qquad (5.10)$$

Here $E_{\text{int.}}$ is the expectation value of the potential energy for the exact ground state wave-functions, Ψ_0, and $\epsilon(k, \Omega)$ is the exact dielectric constant of the plasma. One passes from $E_{\text{int.}}$ to the ground-state energy by means of the formula:

$$E_0 = \int_0^{e^2} dg\, \frac{E_{\text{int.}}(g)}{g} + E_F; \qquad (5.11)$$

the latter expression is simply derived by regarding the charge on the electron as variable (g) and differentiating the expression for the exact ground-state energy.

With the aid of (5.10) we can see directly the role played by the screening action and the collective oscillations on the ground-state energy. First, we note that the expectation value of $E_{\text{int.}}$ in the ground state Ψ_0 appropriate to the RPA is equivalent to taking the expectation value in the *free electron* plane wave state of the following operator

$$H_{\text{coul}}^{\text{eff.}} = \sum_k \frac{2\pi e^2}{k^2} (\rho_k^{\,s})^* \rho_k^{\,s},$$

where the operator $\rho_k^{\,s}$ represents a screened density fluctuation, defined by its matrix elements in the plane wave representation as

$$(\rho_k^{\,s})_{nm} = \frac{(\rho_k)_{nm}}{\epsilon(k, \omega_{nm})}.$$

Thus we see that the RPA result for the interaction energy can be interpreted physically in terms of each electron carrying with it a polarization cloud which modifies its effective charge fluctuation from $e\rho_k$ to $e\rho_k^{\,s}$. The screening is just that appropriate to the dielectric constant at the excitation frequency under consideration, and therefore the above treatment may be regarded as a distinct improvement over the static Fermi-Thomas screening. Second, one can show directly from (5.10) that the zero's of $\epsilon(k, \Omega)$ give rise to a contribution to the ground-state energy which represents the zero-point energy of the plasmons, $\sum_{k<k_c} \frac{\hbar\omega}{2}$, as might have been anticipated from the fact that the plasmons constitute an independent mode of excitation for $k < k_c$.

With the aid of (5.2), (5.10), and (5.11), or by direct calculation (GELL-MANN and BRUECKNER, 1957), one can show that in the RPA the ground-state energy per electron is given by

$$E_0 = \frac{2{\cdot}21}{r_s^{\,2}} - \frac{0{\cdot}916}{r_s} + 0{\cdot}062 \ln r_s - 0{\cdot}142 + 0(r_s) \text{ ryd.} \quad (5.12)$$

The first two terms represent the kinetic energy per electron, and the exchange energy arising as a consequence of the Pauli principle. These are the only terms present in a calculation made with the free electron wave-functions, that is, in the Hartree-Fock approximation. The remaining terms, known as the correlation energy (because they arise as a consequence of including dynamic correlations in the positions of the electrons due to the Coulomb interaction), represent the corrections to the ground-state energy arising within the RPA. GELL-MANN and BRUECKNER (1957) have shown that the corrections to the RPA give rise to terms in the correlation energy of the form

$$A + Br_s + Cr_s \ln r_s + Dr_s^{\,2} + \ldots \quad (5.13)$$

Of the constants appearing in (5.13), only A has thus far been calculated. It is clear from the form of this expansion that the parameter which measures the success of the RPA, and which enables a perturbation-theoretic approach to the quantum plasma is r_s, the inter-electron spacing measured in units of the Bohr radius. This parameter differs slightly from its classical analogue as a consequence of exchange effects. In the quantum case, one finds that the ratio of Coulomb potential energy to kinetic energy is, as one might expect by analogy to the classical case,

$$\frac{e^2 k_{FT}}{E_F} \sim (k_{FT} r_0)^3 \sim r_s^{3/2}.$$

The quantum expansion parameter differs from this because what enters first is not the classical Coulomb energy, $e^2 k_{FT}$, but the exchange energy, e^2/r_0, so that it is r_s, the ratio of exchange energy to kinetic energy, which is the relevant parameter.

In contrast to the classical case, the foregoing simple theory of a quantum plasma is not correct for the domain of greatest physical interest, that of electron densities of the order of those encountered in actual metals. The above considerations lead one to expect that the RPA theory of the quantum plasma will be valid provided $r_s \leqslant 1$, and more detailed considerations (NOZIÈRES and PINES, 1958b) bear out this expectation. For electrons in metals one has

$$1{\cdot}8 \leqslant r_s \leqslant 5{\cdot}5$$

so that a correct theory of metallic plasmas requires an 'intermediate coupling' theory of plasma behaviour, in contrast to the 'weak coupling' theory we have been discussing. Thus the RPA quantum plasma is at best a highly idealized model for a metal, corresponding as it does to an electron density at least ten times higher than that encountered in actual metals.

The situation is not, however, completely hopeless, for it seems possible to devise reasonable interpolation procedures which enable one to discuss plasma behaviour at metallic densities. The weak coupling expansion requires the use of the RPA for all momentum transfers k present in the Coulomb interactions. As r_s increases, the RPA begins to fail; however it fails first for the short wavelengths ($k \gtrsim k_0$). An investigation of the validity of the RPA shows that for metallic densities the RPA yields a reasonable approximation for the dielectric constant, $\epsilon(k, \Omega)$,

VOL 2 196

as long as $k \leqslant k_c \cong 0{\cdot}47 r_s^{1/2} k_0$. For wave-vectors large compared to k_c, a different approximate calculation of the contribution to the ground state energy can be carried out, so that the interpolation procedure involves an interpolation in the contributions to the ground state energy for intermediate values of the wave-vector, of the order of k_c. HUBBARD (1957 and 1958) and NOZIÈRES and PINES (1958b) have proposed closely related methods of carrying out such interpolations; the accuracy of the results is of the order of 15 per cent.

6

The theory we have described in the preceding section provides a good qualitative and, in many cases, quantitative account of the experimental observations of the electronic excitation spectrum in solids. When a beam of fast electrons is sent through, or reflected from, a thin solid film it is found that well-defined peaks tend to appear in the characteristic energy loss spectrum, peaks which correspond to electrons which have given up energy in multiples of a basic quantum, ΔE. For a wide variety of solids ΔE is very nearly equal to $\hbar\omega_p$, the energy of a plasmon, computed under the approximation that all the valence electrons of the solid are free. At first sight this is a surprising result, since one might expect that the binding of the electrons in the periodic potential of the ion cores would produce significant changes in $\epsilon(k, \Omega)$ and in the plasmon energy. However, as we shall see, the plasmon energies are frequently so large compared to the important inter-band or intra-band excitation energies that little change in $\hbar\omega_p$ will occur from a free-electron theory for many solids.

The RPA calculation of $\epsilon(k, \Omega)$ is straightforward (NOZIÈRES and PINES, 1959). It yields a result identical in form with (5.2):

$$\epsilon(k, \Omega) = 1 + \frac{4\pi e^2}{m}\left\{\sum_\mu \frac{f_{0\mu}}{\omega_{\mu 0}^2 - \Omega^2} + i\pi \sum_\mu \frac{f_{0\mu}}{\omega_{\mu 0}} \delta(\omega_{\mu 0} - \Omega)\right\} \qquad (6.1)$$

where now the $f_{0\mu}$ and $\omega_{\mu 0}$ are the single-electron Bloch wave oscillator strengths and excitation frequencies appropriate to the valence electrons moving in the periodic potential field of the ions. In the approximation in which the recoil of the fast charged particle may be neglected, one may apply at once a dielectric theory such as that of Section 3 to calculate the probability per unit time that the particle transfer energy $\hbar\Omega$, and momentum $\hbar\mathbf{k}$, to the solid:

$$\frac{dW}{dt}(k, \Omega) = -\frac{8\pi e^2}{k^2} \Omega \operatorname{Im} \frac{1}{\epsilon(k, \Omega)}. \qquad (6.2)$$

Just as was the case for the classical plasma, there may be both a collective and an individual electron contribution to the energy loss mechanism. The distinction is the same as in Section 3. The collective contribution from the plasmon modes is associated with the zero's of $\epsilon(k, \Omega)$; it will be of importance for those wave-vectors k such that $\epsilon_1(k, \Omega) = 0$, $\epsilon_2(k, \Omega) \ll 1$. The individual electron contribution arises from the finite values of ϵ_2, and appears multiplied by a screening factor of $(\epsilon_1^2 + \epsilon_2^2)^{-1}$.

Inspection of (6.1) shows that in the RPA the plasmon excitation mode will be of importance for those solids for which the majority of the oscillator strengths $f_{0\mu}$ are associated with excitation frequencies, $\omega_{\mu 0}$, which are small compared to the valence electron plasma frequency, ω_p. One can then carry out the high-frequency expansion of ϵ_1,

$$\epsilon_1 \cong 1 - \frac{4\pi e^2}{m} \sum_\mu \frac{f_{0\mu}}{\Omega^2}\left\{1 + \frac{\omega_{\mu 0}^2}{\Omega^2} + \ldots\right\} \qquad (6.3)$$

$$\cong 1 - \frac{\omega_p^2}{\Omega^2} + \ldots$$

where the f-sum rule, $\sum_\mu f_{0\mu} = n$, has been used to obtain the second form of ϵ_1. By definition, $\epsilon_2 \ll 1$ for $\Omega \cong \omega_p$ under the above assumptions. For most metals and many semiconductors the above conditions are well-satisfied; the energy in a plasmon quantum, $\hbar\omega_p$, is of the order of 10–15 eV, while the important intra-band or inter-band oscillator strengths are found at much lower energies (1–5 eV). In Table 1 we reproduce a comparison of calculated valence electron-plasmon energies and the observed characteristic energy losses for a series of solids for which the above approximations might be expected to apply. The agreement is equally good for the alkali metals; for the transition metals, for which the inter-band transition energies are comparable to $\hbar\omega_p$, no such good agreement is observed.

All of the foregoing considerations are based on the assumption that the RPA is valid for the calculation of the long wavelength dielectric constant, $\epsilon(k, \Omega)$, of

TABLE 1.—A COMPARISON OF $\hbar\omega_p$ WITH THE OBSERVED CHARACTERISTIC ENERGY LOSS, ΔE, FOR SOLIDS IN WHICH THE VALENCE ELECTRONS ARE WEAKLY BOUND
(Z denotes the number of valence electrons per atom we have assumed take part in plasma oscillation)

Element	Be	B	C	Mg	Al	Si	Ge
Z	2	3	4	2	3	4	4
$\hbar\omega_p$ (eV)	19	24	25	11	16	17	16
ΔE_{obs} (eV)	19	19	22	10	15	17	17

the valence electrons in solids. No detailed investigations have been carried out on this point. Qualitative considerations (NOZIÈRES and PINES, 1959) indicate that it may be expected to provide a fairly good approximation for essentially the same reasons that it is successful for actual metallic densities in the case of the free-electron gas. Departures from the RPA at long wavelengths arise as a consequence of the coupling between the one-electron excitation modes and the plasmon modes; this coupling is of order $\omega_{\mu 0}^2/\omega_p^2$, and is small for the 'well-behaved' valence electron solids we have been discussing. NOZIÈRES and PINES (1959) have pointed out that a careful comparison of optical data with characteristic energy loss data for a given solid will provide an experimental check on the validity of the RPA for the solid in question.

7

The theory of the classical and quantum plasma may be easily generalized to take into account the motion of the ions within the RPA. One accomplishes this for the classical plasma by adding to the dielectric constant, $\epsilon(k, \Omega)$, of (3.13), the free ionic polarizability,

$$4\pi\alpha_{\text{ion}}(k, \Omega) = + \frac{4\pi e^2}{M_{\text{ion}} k^2} \int d^3\mathbf{v} \frac{\mathbf{k} \cdot \nabla_{\mathbf{v}} f_0^{\text{ion}}(\mathbf{v})}{\Omega - \mathbf{k} \cdot \mathbf{V} + i/\tau}. \quad (7.1)$$

The solutions of the generalized dispersion relation may then be investigated in straightforward fashion. One finds that for a sufficiently large temperature difference between the electrons and ions there is a new low-frequency collective mode, corresponding to oscillations of the ions plus their associated electron screening clouds. At long wavelengths the dispersion relation for the low-frequency modes is of a sound-wave type,

$$\omega \cong sk \cong \sqrt{\frac{m}{M}} v_- k = \sqrt{\frac{T_-}{T_+}} v_+ k \quad (7.2)$$

The long-wavelength oscillations are relatively undamped for an electron-ion temperature ratio, $T_-/T_+ \gtrsim 10$. As the temperature ratio decreases, the damping of the sound waves by the individual ionic motion plays an important role, so that for $T_-/T_+ \approx 3$, there is little or no trace of low-frequency collective motion.

For the quantum plasma of electrons in a metal, it is again easy to include the ion motion within the RPA, within the approximation of considering only the Coulomb interaction between the electrons and the ions (PINES, 1959). A rather good approximation, which neglects slight effects of the periodicity associated with the ions arrayed in a lattice, is to simply add to the quantum dielectric constant, (5.2), an ionic polarizability

$$4\pi\alpha_{\text{ion}} \cong -\frac{\Omega_p^2}{\Omega^2}. \quad (7.3)$$

One then finds the low-frequency collective mode possesses a dispersion relation in the long-wavelength limit which is

$$\omega \cong \sqrt{\frac{mZ}{3M}} v_0 k; \quad (7.4)$$

this provides a good order of magnitude estimate of longitudinal sound wave velocities in actual metals, as was first shown by STAVER (1952). Again note the close relationship between the 'classical' sound velocity in (7.2) and the quantum sound velocity in (7.4). There is a slight damping of the low-frequency quantum sound waves due to individual electron motion, which may be calculated from (5.2) and (7.3), and which turns out to be of order $\sqrt{m/M}$.

One can further apply these results to derive in simple fashion the effective interaction between the electrons in a metal, including the effects of phonon exchange (PINES, 1959). The latter provides the starting point for the BCS theory of superconductivity (BARDEEN *et al.*, 1957). As ANDERSON (1958) has shown the BCS theory combined with the RPA may be regarded as a generalized RPA theory of electron-ion motion, in which the possibility of having a quite different ground state wave function due to the net attractive electron-electron is taken into account.

Indeed, perhaps the most challenging problem which remains in the study of the quantum electron-ion plasma is the justification (in the same sense that it has proved possible to justify the RPA for the simple electron quantum plasma) of the extended random phase approximation in the theory of superconductivity.

REFERENCES

ANDERSON P. W. (1958) *Phys. Rev.* **112,** 1900.

BARDEEN J., COOPER L. and SCHRIEFFER J. R. (1957) *Phys. Rev.* **108,** 1175.

BOHM D. (1959) in *The Many-body Problem,* Dunod, Paris.

BOHM D. and GROSS E. P. (1949) *Phys. Rev.* **75,** 1851, 1864.

BOHM D. and PINES D. (1952) *Phys. Rev.* **85,** 338.

BOHM D. and PINES D. (1953) *Phys. Rev.* **92,** 609.

COHEN M. and EHRENREICH H. To be published.

DUBOIS D. (1959) *Ann. Phys.* **7,** 174.

FERRELL R. (1957) *Phys. Rev.* **107,** 450.

GELL-MANN M. and BRUECKNER K. (1957) *Phys. Rev.* **106,** 364.

HUBBARD J. (1955) *Proc. Phys. Soc.* **A68,** 916.

HUBBARD J. (1957) *Proc. Roy. Soc.* **A240,** 539; (1958) *ibid.* **A243,** 336.

KLIMONTOVICH I. U. L. and SILIN V. P. (1952) *Soviet. Phys. JETP* **23,** 151.

LANDAU L. D. (1946) *J. Phys. U.S.S.R.* **10,** 25.

LINDHARD J. (1954) *Kgl. Danske Videnskab. Mat-fys. Medd.* **28,** No. 8.
LIPKIN H. (1959) *Phys. Rev.* (*Letters*) **2,** 159.
LORENTZ H. A. (1952) *The Theory of Electrons*, Dover Publications, New York.
NOZIÈRES P. and PINES D. (1958a) *Nuovo Cimento* **9,** 470.
NOZIÈRES P. and PINES D. (1958b) *Phys. Rev.* **111,** 442.
NOZIÈRES P. and PINES D. (1959) *Phys. Rev.* **113,** 1254.
PINES D. (1959) in *The Many-body Problem*, Dunod, Paris.
ROSTOKER N. and ROSENBLUTH M. N. *Phys. Fluids* To be published.
SAWADA K. (1957) *Phys. Rev.* **106,** 372.
SAWADA K. BRUECKNER K., FUKUDA N. and BROUT R. (1957) *Phys. Rev.* **108,** 507.
STAVER T. (1952) Ph.D. Thesis, unpublished. Princeton University.
VLASOV A. (1945) *J. Phys. U.S.S.R.* **9,** 25.

VOL.
2
1961

J. Nucl Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 18 to 24. Pergamon Press Ltd. Printed in Northern Ireland

QUASI-PARTICLE APPROACH TO INTERACTION IN AN IDEALIZED METAL*

J. J. QUINN† and R. A. FERRELL‡
University of Maryland, College Park, Maryland

Abstract—The properties of a degenerate electron gas with a background of positive charge capable of propagating phonons are studied by means of a quasi-particle approach. The ground state of the system is pictured as a 'vacuum' state, and any additional particles or holes together with their polarization clouds are thought of as quasi-particles. We treat separately the Coulomb repulsion and the effective electron-electron attraction due to phonon exchange. The specific heat, spin susceptibility and compressibility are determined in terms of the effective energy of interaction $V_{\mathbf{pp}'}$ of two quasi-particles. Considering only the lowest order process contributing to $V_{\mathbf{pp}'}$ reproduces known results for the high density limit Coulomb correction to the specific heat and spin susceptibility, but gives a new result for the phonon correction. To lowest order it is found that the electron-phonon interaction produces no change in the spin susceptibility, in contrast to general expectation. For real metallic densities higher order graphs must be included, however for this case it may be possible to choose $V_{\mathbf{pp}'}$ empirically.

1. INTRODUCTION

THE idealized model for a metal to be used in the present paper is that of a degenerate electron gas together with a background of positive charge capable of propagating sound waves. The effective interaction between electrons then consists of two parts, the Coulomb repulsion and an effective attractive interaction resulting from the virtual exchange of phonons. The two parts of the interaction shall be treated separately.

The states of the system which shall be considered are the ground state and certain simple excited states which differ from the ground state by the excitation of only a small fraction of the total number of particles. For such states considerable simplification can be attained by comparing the ground state of the system to the vacuum state in quantum electrodynamics (QUINN and FERRELL, 1958). The simple excited states mentioned above can then be described in terms of the 'vacuum' state together with a small number of particle-hole pairs. As usual a hole corresponds to a vacancy in a negative energy state and a particle to a filled positive energy state (energy is measured relative to the Fermi energy). In further analogy with electrodynamics, the electron gas is a dielectric medium with definite electrical properties (QUINN and FERRELL, 1958; LINDHARD, 1954) described by a frequency and wave-number dependent dielectric constant. A single positive energy electron (or a hole) will polarize this dielectric medium. The single electron (or hole) together with the polarization cloud it induces around itself can be thought of as a quasi-particle (LANDAU, 1957; QUINN and FERRELL, 1959). The interaction of an electron or hole with its own polarization cloud (QUINN and FERRELL, 1958) can be described as the difference between the self-energies of the quasi-particle and the bare particle. This difference can be thought of as a mass renormalization. Thus because of the polarization cloud, quasi-particles have an effective mass $m^* \neq m$, and they interact with one another through a screened rather than a bare interaction.

It has been shown (QUINN and FERRELL, 1958; HUGENHOLTZ, 1957) that single particle excited states damp out with a mean life which is proportional to $(p - p_0)^{-2}$, where p is the momentum of the excited particle and p_0 the Fermi momentum. This damping is a consequence of the fact that single-electron excitation is not a stationary state of the combined system of electron and degenerate gas and results from inelastic collisions of the original electron with the electrons in the gas. In this paper we limit our consideration to quasi-particles very close to the Fermi surface so that the damping is negligible. Only in this case does it seem reasonable to talk about quasi-particles at all. It is tacitly assumed throughout the following work that the true eigenstates of the system can be put into one to one correspondence with the eigenstates of the non-interacting system, so that if the

* This work was supported in part by the Office of Naval Research under Contract Nonr 1797(00) and by the Air Force Office of Research and Development under Contract AF 49-638-399 with the University of Maryland.
† Present address; RCA Laboratories, Princeton, New Jersey.
‡ Present address: CERN, Geneva, Switzerland.

interaction is slowly turned on, the unperturbed states evolve continuously into the true eigenstates. This assumption is certainly reasonable when we limit ourselves to excited states of very long lifetimes, where it is meaningful to consider single-particle excitations.

2. EFFECTIVE MASS AND SPECIFIC HEAT

Consider the hypothetical excited state of an N-electron system in which just one quasi-particle is present. The energy of this state, relative to 'vacuum', can be expressed as $\epsilon(p, p_0)$, the energy of a single quasi-particle (QUINN and FERRELL, 1958) of momentum p in a degenerate electron gas of Fermi momentum p_0. This state is not a true excited state of the N-particle system, since we have introduced one extra electron (if $p > p_0$) but no holes. It is useful to consider such a state however, for the energy of a true excited state can be expressed as a sum of the individual energies of the quasi-particles plus a term representing their interaction with one another.

Consider the excited state consisting of a single quasi-particle right at the Fermi surface as shown in Fig. 1(a). The energy and momentum of this state are given by

$$E = [\epsilon(p,p_0)]_{p=p_0}, \tag{1}$$

$$\mathbf{P} = \mathbf{p}_0. \tag{2}$$

Place an observer in a frame of reference moving with a velocity $\mathbf{v}$, where the direction of $\mathbf{v}$ is opposite to $\mathbf{P}$ and $|\mathbf{v}| \ll \frac{p_0}{m}$. Define an increment in momentum Δp equal to mv. By a simple Galilean transformation (LANDAU, 1957; QUINN and FERRELL, 1959) the energy in the moving frame can be obtained, and it is given by

$$E' = [\epsilon(p,p_0)]_{p=p_0} + \frac{p_0}{m}\Delta p + (N+1)\frac{mv^2}{2}. \tag{3}$$

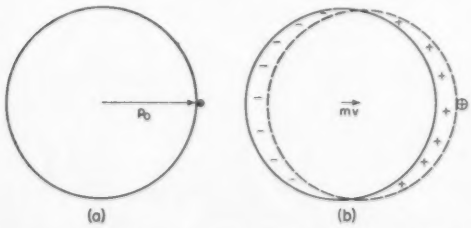


FIG. 1.—The state with one quasi-particle at the Fermi surface is represented in (a). The plus sign enclosed in the small circle represents the quasi-particle; the large circle represents the Fermi surface. When the observer moves to the left with a velocity v, this state can be pictured as shown in (b). In the observer's frame of reference the distribution is centred about the point mv in momentum space, but this state can be pictured in terms of a distribution centred about the origin (the 'vacuum' state) plus additional quasi-particles. The plus and minus signs represent respectively the particles and holes created by the transformation.

This state can also be pictured as the 'vacuum' state plus a number of additional quasi-particles near the Fermi surface, as shown in Fig. 1(b). From this picture it is evident that E' is simply the sum of all the single quasi-particle energies plus their energy of interaction. For convenience we keep the terms associated with the original quasi-particle of Fig. 1(a) separate and lump all the rest of the energy into one term E_T:

$$E' = [\epsilon(p,p_0) + \sum_{\mathbf{p}'}\{n^{(+)}(\mathbf{p}')V_{\mathbf{pp}'}^{(+)} + n^{(-)}(\mathbf{p}')V_{\mathbf{pp}'}^{(-)}\}]_{p=p_0+\Delta p} + E_T, \tag{4}$$

where $V_{\mathbf{pp}'}$ is the energy of interaction of two quasi-particles of momenta $\mathbf{p}$ and $\mathbf{p}'$, and

$$n(\mathbf{p}') = \begin{cases} +1 \text{ if the state } p' > p_0 \text{ is occupied by a quasi-particle,} \\ -1 \text{ if the state } p' < p_0 \text{ is occupied by a quasi-particle,} \\ 0 \text{ otherwise.} \end{cases} \tag{5}$$

The plus and minus superscripts on $n(\mathbf{p}')$ and $V_{\mathbf{pp}'}$ refer to the two possible spin states; in equation (4) the plus is taken as the spin of the original quasi-particle in Fig. 1(a). It is clear that two different interaction energies $V_{\mathbf{pp}'}^{(+)}$ and $V_{\mathbf{pp}'}^{(-)}$ are necessary, since only particles of like spin can undergo exchange.

One can easily show that E_T is equal to $Nmv^2/2$, so that from equations (3) and (4) an expression for $[\partial\epsilon(p,p_0)/\partial p]_{p=p_0}$ can be obtained. If we define the effective mass of a quasi-particle at the Fermi surface by the equation

$$\frac{p_0}{m^*} = \left[\frac{\partial\epsilon(p,p_0)}{\partial p}\right]_{p=p_0}, \tag{6}$$

then from equations (3) and (4)

$$\frac{1}{m^*} = \frac{1}{m} - \frac{1}{p_0}\lim_{\Delta p\to 0}\left[\frac{1}{\Delta p}\sum_{\mathbf{p}'}\{n^{(+)}(\mathbf{p}')V_{\mathbf{pp}'}^{(+)} + n^{(-)}(\mathbf{p})V_{\mathbf{pp}'}^{(-)}\}\right]_{p=p_0}. \tag{7}$$

Assume $V_{\mathbf{pp}'}$ depends only on $|\mathbf{p}-\mathbf{p}'|$. If $\mathbf{p}$ and $\mathbf{p}'$ both fall very close to the Fermi surface, $V_{\mathbf{pp}'}$ depends only on the angle between $\mathbf{p}$ and $\mathbf{p}'$. Let μ be the cosine of this angle; converting the sum in equation (7) to an integral and simplifying gives the following formula for the effective mass,

$$\frac{1}{m^*} = \frac{1}{m} - \frac{\Omega p_0}{(2\pi)^2}\int_{-1}^{1} d\mu\ \mu[V^{(+)}(\mu) + V^{(-)}(\mu)], \tag{8}$$

where Ω is the volume of quantization and $V(\mu)$ shows the explicit dependence of $V_{\mathbf{pp}'}$ on angle.

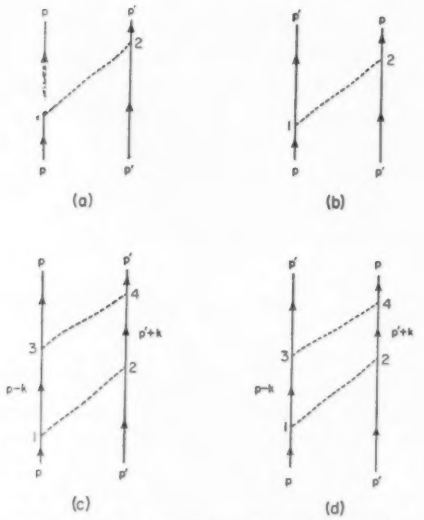


FIG. 2.—The second [(a), (b)] and fourth [(c), (d)] order processes contributing to $V_{pp'}$ are shown. The dashed lines represent the propagation of a charge disturbance or polarization emitted by one quasi-particle and absorbed by the other. Figures (a) and (c) are direct processes, (b) and (d) exchange processes. The second-order direct process, (a), is cancelled out by the background of positive charge and can be omitted. Thus the only second-order process contributing to $V_{pp'}$ is the exchange process (b), which occurs only for particles of like spin. To obtain an exact expression for $V_{pp'}$ the contribution of all processes must be summed to infinite order.

The Sommerfeld specific heat for a non-interacting electron gas is given by

$$C_s = \left(\frac{\pi K}{p_0}\right)^2 m\,T, \tag{9}$$

where K is Boltzmann's constant and T is absolute temperature. By simply replacing m the bare electron mass by m,* one obtains the specific heat of the interacting electron gas, as pointed out by LANDAU (1957).

Equations (8) and (9) have thus reduced the problems of the effective mass and specific heat of a many-body system to a calculation of the energy of interaction of two quasi-particles in the electron-gas vacuum. This calculation resembles the relativistic bound state problems in quantum electrodynamics discussed by BETHE and SALPETER (1951).

Consider first the Coulomb interactions. A few of the lowest order processes contributing to $V_{pp'}$ are shown in Fig. 2. The polarization propagator $G_+(2,1)$ has been discussed in detail elsewhere (QUINN and FERRELL 1958; LINDHARD 1954; DUBOIS, 1959; ROCKMORE, 1959) and is given by

$$G_+(2,1) = \int \frac{d^3k\,d\omega}{(2\pi)^4}\,\frac{4\pi}{k^2\epsilon_0(\mathbf{k},\,\omega)}\exp\left(i\mathbf{k}\cdot\mathbf{x}_{2,1} - i\omega t_{2,1}\right). \tag{10}$$

where $\epsilon_0(\mathbf{k},\,\omega)$ is the wave-number and frequency dependent dielectric constant. For simplicity we shall calculate $V_{pp'}$ to only lowest order. Since the second-order direct interaction is exactly cancelled by the background of positive charge, to this order $V^{(-)}_{pp'}$ must vanish. This occurs because quasi-particles of unlike spin cannot undergo exchange, the only process left which contributes to the energy in lowest order. This approximation is very crude, and one cannot expect results of great accuracy. It can easily be shown that to lowest order

$$V^{(+)}_{pp'} = -\frac{4\pi e^2}{\Omega}\,\frac{1}{|\mathbf{p}-\mathbf{p}'|^2\epsilon_0(|\mathbf{p}-\mathbf{p}'|,\;\epsilon(p)-\epsilon(p'))} \tag{11}$$

Since we are considering only quasi-particles right at the Fermi surface, $\epsilon(p) = \epsilon(p') = \epsilon(p_0)$, and equation (11) involves only the static dielectric constant. To a fairly good approximation the static dielectric constant is given by

$$\epsilon_0(\mathbf{k},0) = 1 + \frac{q^2}{k^2}, \tag{12}$$

where the screening constant q is equal to $(4k_0/\pi a_0)^{1/2}$ (k_0 is the Fermi wave-number and a_0 is the radius of the first Bohr orbit in hydrogen). In terms of the commonly used density parameter r_s, $q = 0{\cdot}815\,k_0 r_s^{1/2}$. From equations (8), (11) and (12) one can easily obtain the result

$$\frac{1}{m^*} = \frac{1}{m} - \frac{e^2}{2\pi p_0}\left\{\ln\left(\frac{\alpha r_s}{\pi}\right) + 2\right\}, \tag{13}$$

where α is a constant equal to $(4/9\pi)^{1/3}$. Replacing m by m^* in equation (9) gives for the specific heat the well-known result of GELL-MANN (1957). Presumably including higher-order graphs in the calculation of $V_{pp'}$ will yield the higher-order corrections to the GELL-MANN result obtained by DUBOIS (1959).

It is interesting to note that the calculation of the effective mass and specific heat involves only a static screened potential. The dielectric constant approach, or alternatively summing certain terms to all orders of perturbation theory leads to a Yukawa potential. Choosing some phenomenological screened potential slightly different from the Yukawa will lead to a slightly different result. Thus surprisingly, the only essential difference between the specific heat calculation of GELL-MANN and that of PINES (1953) is the choice of screened potential. The Yukawa potential is known to give the exact result in the high-density limit but perhaps a different choice of screened potential might give more accurate results in the region of real metallic densities.

For the phonon exchange interaction the processes contributing to $V_{\mathbf{pp}'}$ are the same as those shown in Fig. 2 if the polarization propagator is replaced by a phonon propagator. To lowest order, $V_{\mathbf{pp}'}$ for the phonon exchange interaction is given by

$$V_{\mathbf{pp}'}^{(+)} = \frac{-2\omega_{\mathbf{p}-\mathbf{p}'}|H'_{\mathbf{p}-\mathbf{p}'}|^2}{[\epsilon(p) - \epsilon(p')]^2 - \omega_{\mathbf{p}-\mathbf{p}'}^2} \tag{14}$$

where $|H'_{\mathbf{p}-\mathbf{p}'}|^2$ is the square of the matrix element for phonon excitation and $\omega_{\mathbf{p}-\mathbf{p}'}$ is the phonon frequency.* $|H'_{\mathbf{p}-\mathbf{p}'}|^2$ depends only on the magnitude of $\mathbf{p}-\mathbf{p}'$ and is given by

$$|H'_{\mathbf{p}-\mathbf{p}'}|^2 = \frac{4C^2}{9n\Omega}\frac{|\mathbf{p}-\mathbf{p}'|}{Mc_s}, \tag{15}$$

where M and n are the ionic mass and density,† c_s the velocity of sound and C is BETHE's (1933) constant. The phonon frequency $\omega_{\mathbf{K}}$ is $c_s|\mathbf{K}|$ where $\mathbf{K}$ is limited by its maximum value K_D, the Debye cutoff. As before $\epsilon(p) = \epsilon(p')$ since the quasi-particles considered are very close to the Fermi surface; this leads to the simple result that $V_{\mathbf{pp}'}^{(+)}$, is a constant given by

$$V_{\mathbf{pp}'}^{(+)} = \frac{8C^2}{9n\Omega Mc_s^2}. \tag{16}$$

The single-particle energy now includes a phonon exchange part which does not transform like equation (3) due to the Doppler shift in the phonon frequencies. Therefore equation (8) is no longer applicable. An alternative approach to the calculation of effective mass, which does not depend on Galilean invariance, is the self-energy method (QUINN and FERRELL, 1958). The Feynman graphs for the self-energy process are shown in Fig. 3. The formula for the self-energy is

$$\text{S.E.}(p) = \int d^3x_1\, dt_1 \exp(-i\mathbf{p}\cdot\mathbf{x}_2 + iE(p)t_2) \times K_+(2,1)P(2,1)\exp(i\mathbf{p}\cdot\mathbf{x}_1 - iE(p)t_1), \tag{17}$$

where $K_+(2,1)$ and $P(2,1)$ are the electron and phonon propagators respectively. The electron propagator has been discussed extensively in QUINN and FERRELL (1958); the phonon propagator is given by:

$$P(2,1) = \int \frac{d^3k\, d\omega}{(2\pi)^4}\frac{2\omega_k|H_k'|^2}{\omega^2 - \omega_k^2}\exp(i\mathbf{k}\cdot\mathbf{x}_{2,1} - i\omega t_{2,1}), \tag{18}$$

where ω_k is the phonon frequency and equals $c_s k$. By evaluating the integral in equation (17), adding the kinetic energy and obtaining the derivative with respect to momentum at the Fermi surface one obtains

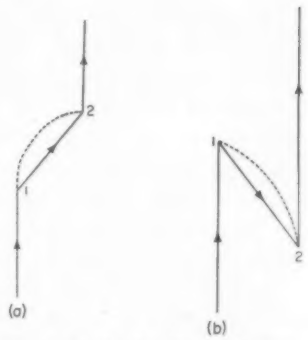


FIG. 3.—Feynman graphs for the self-energy process. In graphs of type (a) the particle in the intermediate state propagates forward in time (flow of time is upward). Inclusion of graphs of type (b) when the particle considered is an additional electron and can propagate backward in time accounts for exchange effects. The process (b) represents the simultaneous creation at a space-time point 2 of a phonon and an electron-hole pair. The hole propagates forward in time to 1 (represented by backward electron propagation) and annihilates with the incident electron absorbing the phonon.

$$\frac{1}{m^*} = \frac{1}{m}(1 - G) \tag{19}$$

where G is essentially Fröhlich's dimensionless coupling constant and is given by

$$G = \frac{1}{6}\left(\frac{K_D}{k_0}\right)^2\frac{C^2}{Mc_s^2E_0}. \tag{20}$$

Comparing this with the more accurate result obtained by using Brillouin-Wigner perturbation theory (FERRELL, 1958)

$$m^* = m(1 + G) \tag{21}$$

shows that equation (19) corresponds to an expansion of $(1 + G)^{-1}$ where only the first two terms are retained. Comparing equation (19) with the result obtained by substituting equation (14) into equation (8)

$$\frac{1}{m^*} = \frac{1}{m}\left[1 - G\left(1 - \left[\frac{K_D}{2k_0}\right]^2\right)\right], \tag{22}$$

shows the importance of the Galilean invariance.

The effective mass given by equations (19) or (21) leads to about a 20 per cent enhancement in the electronic specific heat of sodium. The Coulomb interaction inhibits the specific heat. The high-density result, equation (13), is not applicable to real metals, but using the Yukawa potential without expanding for small r_s gives roughly a 4 to 5 per cent decrease in the specific heat in the region of real metallic densities. PINES' (1953) result is somewhat different and probably more accurate for lower densities; he obtains an 18 per cent decrease in the specific heat of sodium.

* Throughout this paper we take $\hbar$, the reduced Planck's constant, to be unity.

† In order to specify the modes of vibration of the oscillating background of positive charge, it is necessary to associate with it an ionic mass and density.

3. MAGNETIC SPIN SUSCEPTIBILITY

The magnetic spin susceptibility of an electron gas results from a competition between the energy required to create quasi-particles (to flip some spins) and the energy obtained from the interaction of the resulting unpaired spins with the external magnetic field. Suppose a magnetic field H causes $Ns/2$ spin-down electrons to reverse their spin direction, where N is the total number of particles and s is a fraction very small compared to unity. The amount of energy $\Delta E(s)$ required to cause this spin polarization is proportional to Ns^2:

$$\Delta E(s) = \eta N s^2. \tag{23}$$

The energy of interaction of the spin with the magnetic field is simply $Ns\,\beta H$ where β is the magnetic moment of an electron and H is the magnetic field strength. Equilibrium is attained when the two energies are equal; this gives an equation for the fraction s in terms of the magnetic field:

$$s = \beta H/\eta. \tag{24}$$

The susceptibility is simply the magnetization, $Ns\beta/\Omega$, divided by H:

$$\chi = n\beta^2/\eta, \tag{25}$$

where n is the number of particles per unit volume. This reduces the problem to a calculation of the coefficient η defined by equation (23).

The state in which $Ns/2$ spins have been flipped can be thought of as the 'vacuum' state plus two thin shells of quasi-particles as shown in Fig. 4. The outer shell corresponds to extra electrons of one spin, the inner shell to holes of the opposite spin. The polarization energy is simply the sum of the individual quasi-particle energies plus their mutual energy of interaction. Using the quasi-particle methods of the preceding section one can easily show that

$$\Delta E(s) = Ns^2\left\{\frac{p_0^2}{6m^*} + \frac{N}{8}\int d\mu[V^{(+)}(\mu) - V^{(-)}(\mu)]\right\},$$

$$= Ns^2 \cdot \frac{p_0^2}{6}\left\{\frac{1}{m^*} + \frac{\Omega p_0}{(2\pi)^2}\int d\mu[V^{(+)}(\mu) - V^{(-)}(\mu)]\right\}, \tag{26}$$

where m^* is the effective mass determined in the preceding section. The limits on the integral have been omitted for the sake of generality. For the Coulomb interaction the integral extends from -1 to $+1$, but for the phonon exchange interaction the lower limit is $(1 - K_D^2/2k_0^2)$ where K_D is the Debye cut-off.

Consider as before, only the lowest order process

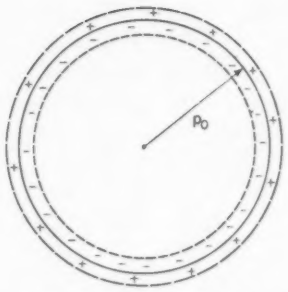


FIG. 4.—The polarization of the 'vacuum' state by a magnetic field results in an excited state consisting of two thin shells of quasi-particles at the Fermi surface. The inner shell consists of holes all of which have the same spin; they are represented by minus signs. The outer shell consists of extra electrons all of the opposite spin from the holes; the extra electrons are represented by plus signs.

contributing to $V_{pp'}$. For the Coulomb interaction this gives for $\Delta E(s)$

$$\Delta E(s) = Ns^2\left\{\frac{1}{3\alpha^2 r_s^2} - \frac{1}{6\pi\alpha r_s}\left(\ln\frac{\alpha r_s}{\pi} + 2\right)\right.$$

$$\left. - \frac{7}{18\pi^2}\ln\frac{\alpha r_s}{\pi} + \frac{1}{6\pi\alpha r_s}\ln\frac{\alpha r_s}{\pi} + \frac{1}{18\pi^2}\ln\frac{\alpha r_s}{\pi} + \ldots\right\} \text{ryd}, \tag{27}$$

where $\Delta E(s)$ is given in Rydbergs for the purpose of comparison with the exact result of BRUECKNER and SAWADA (1958). The first terms come from the effective mass part of equation (26), and the last two terms come from the quasi-particle interactions. These two contributions partially cancel giving

$$\Delta E(s) = Ns^2\left[\frac{1}{3\alpha^2 r_s^2} - \frac{1}{3\pi\alpha r_s} - \frac{1}{3\pi^2}\ln\frac{\alpha r_s}{\pi} + \ldots\right] \text{ryd}. \tag{28}$$

The BRUECKNER-SAWADA result,

$$\Delta E(s) = Ns^2\left[\frac{1}{3\alpha^2 r_s^2} - \frac{1}{3\pi\alpha r_s} - \frac{1}{6\pi^2}\ln\frac{\alpha r_s}{\pi} + \ldots\right] \text{ryd} \tag{29}$$

shows that neglecting the higher-order processes which contributed to $V_{pp'}$ leads to too large a coefficient in the logarithmic term. It is clear that a more accurate calculation of $V_{pp'}$ should give the BRUECKNER-SAWADA result.

For the phonon exchange interaction, the spin susceptibility can be obtained by substituting equation (16) into equation (26). The resulting expression for $\Delta E(s)$ is

$$\Delta E(s) = Ns^2\left(\frac{p_0^2}{6m} - \frac{p_0^2}{6m}G + \frac{p_0^2}{6m}G\right), \tag{30}$$

where again the first two terms are due to the effective mass part of equation (26) and the last term is due to

the quasi-particle interactions. Here the quasi-particle interaction exactly cancels the effective mass correction to the spin susceptibility. As far as the authors are aware this is a new result.

Just as for the effective mass, if the calculation is limited to lowest order processes, only the static screening of the interaction is taken into account. In the spin-susceptibility the corrections due to effective mass and quasi-particle interaction are of opposite sign and tend to cancel; it is just this cancellation which gets rid of the large $r_s^{-1} \ln r_s$ term in the Coulomb correction. It is difficult to estimate the value of the Coulomb correction to the spin susceptibility for real metals since equation (29) is only valid in the high density limit. PINES (1955) has estimated roughly a 30 per cent increase in the spin susceptibility of sodium due to the Coulomb interactions. The phonon exchange interaction gives no contribution to the spin susceptibility for sodium. Thus no strong temperature dependence of the spin susceptibility should be expected.

4. COMPRESSIBILITY

The quantity $[-\epsilon(p,p_0)]_{p=p_0}$ is the amount of energy required to remove one particle from the Fermi surface in an electron gas, and is sometimes called the separation energy (QUINN and FERRELL, 1958), $-E_s(p_0)$. It has been shown elsewhere (QUINN and FERRELL, 1958; HUGENHOLTZ and VAN HOVE, 1958) that the ground-state energy per particle $g(p_0)$ is related to $E_s(p_0)$ by the equation

$$g(p_0) + \frac{p_0}{3} g'(p_0) = E_s(p_0), \tag{31}$$

where the prime denotes differentiation with respect to p_0. The pressure on the electron gas is given by

$$P(p_0) = -\frac{d}{d\Omega}[Ng(p_0)] = \frac{np_0}{3} g'(p_0). \tag{32}$$

The compressibility is defined by the equation

$$\kappa = -\frac{1}{\Omega}\frac{\partial \Omega}{\partial P}, \tag{33}$$

and in terms of the separation energy is given by

$$\kappa = \frac{3}{np_0}\left(\frac{dE_s(p_0)}{dp_0}\right)^{-1}. \tag{34}$$

The problem is therefore to evaluate the derivative of E_s with respect to p_0. From the definition of $E_s(p_0)$ it is clear that

$$\frac{dE_s(p_0)}{dp_0} = \left[\frac{\partial \epsilon(p,p_0)}{\partial p} + \frac{\partial \epsilon(p,p_0)}{\partial p_0}\right]_{p=p_0}. \tag{35}$$

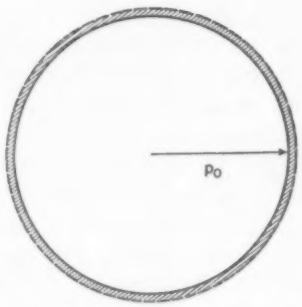


FIG. 5.—The ground state of a system containing N' electrons ($N' > N$ and $(N' - N)/N \ll 1$) can be pictured as the 'vacuum' state plus a thin shell of quasi-particle. As usual the 'vacuum' state is represented by a sphere of radius p_0. The shell is completely occupied by quasi-particles of both spins.

The first term is simply p_0/m^* which was discussed in Section 2.

To evaluate the second term imagine changing the density of the electron gas by increasing the number of electrons from N to N' while holding the volume constant. The ground state of the system of N' electrons can be pictured as the 'vacuum' state of the N-electron system plus a thin shell of quasi-particles at the Fermi surface as shown in Fig. 5. How does $\epsilon(p,p_0)$ change when we add these particles and change p_0 to $(p_0 + \Delta p_0)$? In the N'-electron system a given quasi-particle will interact with the thin shell of quasi-particles at the Fermi surface as well as with the N-electron 'vacuum' state. Therefore the change $\epsilon(p, p_0 + \Delta p_0) - \epsilon(p,p_0)$ in quasi-particle energy is simply the energy of interaction of the given quasi-particle with the thin shell of quasi-particles shown in Fig. 5. This problem is quite similar to the calculation of the spin susceptibility. Using the methods developed in preceding sections gives

$$\left[\frac{\partial \epsilon(p,p_0)}{\partial p_0}\right]_{p=p_0} = \frac{\Omega p_0^2}{(2\pi)^2}\int d\mu[V^{(+)}(\mu) + V^{(-)}(\mu)], \tag{36}$$

where the limits on the integral are exactly the same as for the spin susceptibility discussed in the preceding section. Therefore the derivative of the separation energy is

$$\frac{dE_s(p_0)}{dp_0} = p_0\left\{\frac{1}{m^*} + \frac{\Omega p_0}{(2\pi)^2}\int d\mu[V^{(+)}(\mu) + V^{(-)}(\mu)]\right\}. \tag{37}$$

Comparison with equation (26) shows that except for a difference in sign in the coefficient of the $V_{pp'}^{(-)}$, in the two equations, the spin susceptibility and the compressibility would be proportional. To the order in which we have been calculating $V_{pp'}^{(-)}$, vanishes; thus

to this order χ is proportional to κ. Since the results for χ to this order of approximation have already been given, we shall not repeat them for κ. A comparison of experimental results for χ and κ should indicate just how important the $V_{pp'}^{(-)}$, term is.

By integrating with respect to p_0, in equation (37) one can obtain the separation energy as a function of p_0 up to an arbitrary constant. This constant has been evaluated (QUINN and FERRELL, 1958) by comparison with the GELL-MANN-BRUECKNER (1957) result for the correlation energy. Since the quasi-particle approach avoids the constant, it might be an easier method for calculating the next higher order correction beyond the constant term.

The calculation of the compressibility given above is not applicable to the phonon-electron interaction in real metals. The reason for this is the fact that compressing a metal changes the phonon spectrum and hence the electron-phonon interaction. Therefore equation (36) and (37) are applicable only to the Coulomb interactions, which do not change as the metal is compressed.

5. SUMMARY

The concept of quasi-particle as used in this paper is based upon the analogy between the ground state of a degenerate electron gas and the vacuum state in quantum electrodynamics. Any changes in this ground state can be described in terms of extra electrons in normally unoccupied states and holes in normally occupied states. These extra particles and holes together with their polarization clouds are referred to as quasi-particles. Since a quasi-particle consists of a real particle plus a polarization cloud it is evident that quasi-particles will interact through a screened interaction. In this paper we consider both Coulomb interaction and the effective attractive interaction between electrons due to phonon processes. We have shown how the effective mass, specific heat, spin susceptibility, compressibility and separation energy can be obtained by considering the mutual interactions of quasi-particles. Keeping only the lowest-order process contributing to the quasi-particle interaction energy gives qualitatively correct results, but for greater accuracy higher-order terms must be considered. In the case of real metals it might be possible to choose the $V_{pp'}$ function empirically and thus describe many metallic properties in terms of one empirically determined function.

Acknowledgement—In conclusion one of us (J. J. Quinn) would like to acknowledge several stimulating discussions with DR. R. H. PARMENTER.

REFERENCES

BRUECKNER K. and SAWADA K. (1958) *Phys. Rev.* **112,** 328.
DUBOIS D. F. (1959) *Ann. Phys.* **7,** 174.
FERRELL R. A. (1958) *Bull. Amer. Phys. Soc.* Ser. II **3,** 203.
FRÖHLICH H. (1950) *Phys. Rev.* **79,** 845.
GELL-MANN M. and BRUECKNER K. (1957) *Phys. Rev.* **106,** 369.
HUGENHOLTZ N. M. (1958) *Physica s'Grav.* **23,** 481.
HUGENHOLTZ N. M. and VAN HOVE L. (1958) *Physica s'Grav.* **24,** 363.
LANDAU L. D. (1957) *Soviet Phys. JETP* **3,** 920.
LINDHARD J. (1954) *Kgl. Danske Videnskab. Mat.-fys. Medd.* **28,** No. 8.
PINES D. (1953) *Phys. Rev.* **92,** 626.
PINES D. (1955) *Solid State Physics* (Edited by SEITZ F. and TURNBULL D.) Vol. 1, p. 367, Academic Press, New York.
QUINN J. J. and FERRELL R. A. (1958) *Phys. Rev.* **112,** 812.
QUINN J. J. and FERRELL R. A. (1959) *Bull. Amer. Phys. Soc.* Ser. II **4,** 276.
ROCKMORE R. M. (1959) *Phys. Rev.* **114,** 941.
SALPETER E. E. and BETHE H. A. (1951) *Phys. Rev.* **84,** 1232.
SOMMERFELD A. and BETHE H. (1933) *Handbuch der Physik* **24,** 2, 517.

VOL 2 196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 25 to 26. Pergamon Press Ltd. Printed in Northern Ireland

THE TOMONAGA METHOD AND SOME OF ITS APPLICATIONS

D. TER HAAR
The Clarendon Laboratory, Oxford

Abstract—A brief account is given of TOMONAGA's method of introducing collective co-ordinates. This method is applied to a derivation of the dispersion relations for plasma oscillations. The result is different from the one obtained by PINES and BOHM and possible reasons for this difference are given. Finally we sketch JEPSEN's method for deriving the hydrodynamic equations of motion.

IN 1955 TOMONAGA developed an extremely elegant method to deal with collective phenomena. His method was developed in quantum mechanical terms, but can easily well be stated classically (TER HAAR, 1957 and 1958), as we shall do in this paper.

If some collective motion of the particles in a system can be derived from a displacement potential ϕ, the corresponding collective co-ordinate ξ and collective momentum π will satisfy the equations

$$\xi = A\Sigma_i \phi(\mathbf{x}_i), \tag{1}$$

$$\pi = B\Sigma_i(\mathbf{p}_i \cdot \nabla_i \phi), \tag{2}$$

where the summation is over all particles in the system, where the $\mathbf{x}_i$ and $\mathbf{p}_i$ are the Cartesian co-ordinates and momenta of the particles, and where the A and B are constants which are to be adjusted in such a way that the Poisson bracket (P.B.) of ξ and π is equal to unity:

$$\{\xi,\pi\} = \sum_i \left\{ \left(\frac{\partial \xi}{\partial \mathbf{x}_i} \cdot \frac{\partial \pi}{\partial \mathbf{p}_i} \right) - \left(\frac{\partial \xi}{\partial \mathbf{p}_i} \cdot \frac{\partial \pi}{\partial \mathbf{x}_i} \right) \right\} = 1. \tag{3}$$

The collective kinetic energy T_c is assumed to be quadratic in π,

$$T_c = \tfrac{1}{2} C\pi^2, \tag{4}$$

and the constant C is determined from the condition

$$\{T_{\text{in.}}, \xi\} = \{T - T_c, \xi\} = 0, \tag{5}$$

which expresses the fact that the 'internal' kinetic energy $T_{\text{in.}}$ does no longer contain π. We have now obtained the Hamiltonian of the system in the form

$$H = \tfrac{1}{2} C\pi^2 + \bar{H}, \; \bar{H} = T_{\text{in.}} + U, \tag{6}$$

where $\bar{H}$ still contains ξ as well as the particle co-ordinates.

The next step in TOMONAGA's treatment consists in assuming that in many cases $\bar{H}$ will contain ξ in the form of a quadratic expression. One therefore expands $\bar{H}$ in a Taylor series in ξ,

$$\bar{H} = \bar{H}(0) + \bar{H}'(0)\xi + \tfrac{1}{2}\bar{H}''(0)\xi^2, \tag{7}$$

where the primes indicate differentiation with respect to ξ. These derivatives are now obtained by observing that they can be expressed in terms of P.B.:

$$\bar{H}'(\xi) = \{\bar{H},\pi\}, \quad \bar{H}'' = \{\{H,\pi\},\pi\}, \tag{8}$$

$$\bar{H}(0) = \bar{H}(\xi) - \{\bar{H}(\xi),\pi\}\xi + \tfrac{1}{2}\{\{\bar{H}(\xi),\pi\}\pi\}\xi^2,$$

$$\bar{H}'(0) = \{\bar{H}(\xi),\pi\} - \{\{\{\bar{H}(\xi),\pi\},\pi\}\xi, \tag{9}$$

$$\bar{H}''(0) = \{\{\bar{H}(\xi),\pi\},\pi\},$$

where ξ and π are expressed in terms of the $\mathbf{x}_i$ and $\mathbf{p}_i$, and where equations (9) are valid up to terms in ξ^2.

In many cases the term in ξ in (7) will vanish and in the approximation studied here we have then

$$H_c = H + \bar{\bar{H}}, \qquad \bar{\bar{H}} = \bar{H}(0), \tag{10}$$

where

$$H_c = \tfrac{1}{2} C\pi^2 + \tfrac{1}{2} D\xi^2. \tag{11}$$

From the Hamiltonian equations of motion we then get

$$\dot{\pi} = -\frac{\partial H}{\partial \xi} = -D\xi, \qquad \dot{\xi} = \frac{\partial H}{\partial \pi} = C\pi, \tag{12}$$

and we obtain for the frequency ω of the collective oscillation

$$\omega^2 = CD. \tag{13}$$

In the case of the plasma oscillations it is more convenient to use a complex function $\phi_{\mathbf{k}}$ for the oscillation with wave vector $\mathbf{k}$, and we get instead of equations (1), (2), and (4) (N is the number of particles per unit volume).

$$\xi_{\mathbf{k}} = (Nk)^{-1}\Sigma_j \exp[-i(\mathbf{k} \cdot \mathbf{x}_j)], \tag{14}$$

$$\pi_{\mathbf{k}} = (i/k)\,\Sigma_j(\mathbf{k} \cdot \mathbf{p}_j) \exp i(\mathbf{k} \cdot \mathbf{x}_j), \tag{15}$$

$$T_c = \pi_{\mathbf{k}}\pi_{-\mathbf{k}}/Nm. \tag{16}$$

The potential energy U is now given by the Coulomb interactions between the particles, and using a Fourier expansion in a unit volume we have

$$U = \tfrac{1}{2}\Sigma_{j \neq n}\Sigma_{\mathbf{k}}(4\pi e^2/k^2) \exp i(\mathbf{k} \cdot \mathbf{x}_j - \mathbf{x}_n), \tag{17}$$

while $T_{\text{in.}}$ is given by the equation

$$T_{\text{in.}} = \Sigma_j(\mathbf{p}_j^2/2m) - (Nmk^2)^{-1}|\Sigma_j(\mathbf{k} \cdot \mathbf{p}_j) \exp i(\mathbf{k} \cdot \mathbf{x}_j)|^2. \quad (18)$$

To find the dispersion relations we first of all have to evaluate $\{U, \pi_{\mathbf{k}}\}$ and $\{\{U,\pi_{\mathbf{k}}\}, \pi_{-\mathbf{k}}\}$. The first P.B. can be neglected to a fair approximation and for the second P.B. we have

$$\{\{U, \pi_{\mathbf{k}}\}, \pi_{-\mathbf{k}}\} = 4\pi N^2 e^2. \quad (19)$$

Secondly we must evaluate $\{T_{\text{in.}}, \pi_{\mathbf{k}}\}$ and $\{\{T_{\text{in.}}, \pi_{\mathbf{k}}\}, \pi_{-\mathbf{k}}\}$ for which we get

$$\{T_{\text{in.}}, \pi_{\mathbf{k}}\} = (mk)^{-1}\Sigma_j(\mathbf{k} \cdot \mathbf{p}_j)^2 \exp i(\mathbf{k} \cdot \mathbf{x}_j), \quad (20)$$

$$\{\{T_{\text{in.}}, \pi_{\mathbf{k}}\}, \pi_{-\mathbf{k}}\} = (3/m)\Sigma_j(\mathbf{k} \cdot \mathbf{p}_j)^2. \quad (21)$$

From the Hamiltonian equations of motion we now get instead of (13)

$$\omega_{\mathbf{k}}^2 = (3/Nm^2)\Sigma_j(\mathbf{k} \cdot \mathbf{p}_j)^2 + (4\pi Ne^2/m), \quad (22)$$

or

$$\omega_{\mathbf{k}}^2 = \omega_p^2 + k^2 \langle v^2 \rangle, \quad (23)$$

where

$$\omega_p^2 = 4\pi Ne^2/m, \ \Sigma_j(\mathbf{k} \cdot \mathbf{p}_j)^2 = \tfrac{1}{3}Nk^2\langle v^2 \rangle. \quad (24)$$

Equation (22) differs from the one derived by PINES and BOHM (1952) by the factor 3 in the first term on the right-hand side. It is of interest to note first of all that VAN KAMPEN (1957) has drawn attention to the fact that this factor should be expected whenever all collisions (or interactions) are neglected. Secondly we note that the PINES and BOHM result is obtained if we perform in (20) their averaging process, and write

$$\{T_{\text{in.}},\pi_{\mathbf{k}}\} \doteq (k/3m)\ \langle v^2 \rangle\ \Sigma_j \exp i(\mathbf{k} \cdot \mathbf{x}_j). \quad (25)$$

It would be of interest to discuss in somewhat more detail under what circumstances (25) is a physically justified approximation. As long as the particle motions are nearly independent (VAN KAMPEN's case of no collisions) it is definitely a bad approximation and our derivation must be preferred. If, however, there is such a strong correlation between the particles, that their velocities are governed by their interactions and that one may neglect the random character of these velocities, (25) may well be a good approximation. In that case, of course, the second term in (22) which arises from the potential energy will by far be the dominant one.

In conclusion we wish to show how one can use the TOMONAGA variables $\xi_{\mathbf{k}}$ to derive the hydrodynamical equations of motion (JEPSEN, to be published).

Let $f(\mathbf{x},\mathbf{v},t)$ be the distribution function of the particles in the system, and let $n(\mathbf{x})$ be the number density:

$$n(\mathbf{x}) = \int f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v}. \quad (26)$$

Introduce an average fluid velocity $\mathbf{u}(\mathbf{x})$ by

$$n(\mathbf{x})\mathbf{u}(\mathbf{x}) = \int \mathbf{v} f(\mathbf{x},\mathbf{v},t)\, d^3\mathbf{v}. \quad (27)$$

From equations (14) and (15) we get, using $f(\mathbf{x},\mathbf{v},t)$ and equations (26) and (27)

$$\xi_{\mathbf{k}} = (Nk)^{-1} \int n(\mathbf{x}) \exp[-i(\mathbf{k} \cdot \mathbf{x})]\, d^3\mathbf{x}, \quad (28)$$

$$\pi_{\mathbf{k}} = (i/k) \int mn(\mathbf{x})(\mathbf{u}(\mathbf{x}) \cdot \mathbf{k}) \exp i(\mathbf{k} \cdot \mathbf{x})\, d^3\mathbf{x}, \quad (29)$$

or, integrating by parts

$$\pi_{\mathbf{k}} = -(m/k) \int \exp i(\mathbf{k} \cdot \mathbf{x})(\nabla \cdot n(\mathbf{x})\mathbf{u}(\mathbf{x}))\, d^3\mathbf{x}. \quad (30)$$

From the canonical equations of motion we get now

$$\int \exp\{[-i(\mathbf{k} \cdot \mathbf{x})]\{\dot{n}(\mathbf{x}) + (\nabla \cdot n(\mathbf{x})\mathbf{u}(\mathbf{x}))\}\, d^3\mathbf{x} = 0, \quad (31)$$

and if this equation be true for a complete set of wave vectors $\mathbf{k}$ we get the hydrodynamic equation of continuity:

$$\frac{\partial n}{\partial t} + (\nabla \cdot n\mathbf{u}) = 0. \quad (32)$$

In a similar way one can derive, as was shown by JEPSEN (to be published), the hydrodynamical equation of motion. It is of interest to note that JEPSEN derived the general equation of motion and not the linearized one. We refer to his paper for a discussion of this equation as well as for a discussion of the quantum mechanical case.

REFERENCES

HAAR D. TER (1957) *Rep. Progr. Phys.* **20**, 130; (1958) *Introduction to the Physics of Many-body Systems*, Interscience, New York.

JEPSEN D. W. *Phys. Fluids.* To be published.

KAMPEN N. G. VAN (1957) *Physica s'Grav.* **23**, 641.

PINES D. and BOHM D. (1952) *Phys. Rev.* **85**, 338.

TOMONAGA S. (1955) *Progr. Theor. Phys., Osaka* **13**, 467, 482.

VOL 2 196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 27 to 45. Pergamon Press Ltd. Printed in Northern Ireland

THERMODYNAMIC FUNCTIONS OF A PARTIALLY DEGENERATE, FULLY IONIZED GAS*

H. E. DEWITT
Lawrence Radiation Laboratory, University of California, Livermore, California

Abstract—The formalism of GLASSGOLD, HECKROTTE and WATSON for the expansion of the quantum-mechanical grand partition function has been applied to a gas of point charges. The expansion is shown to be closely related to Rayleigh–Schroedinger perturbation theory, and to give the results of MONTROLL and WARD in a simpler form. The theory has been extended to a multicomponent gas of fermions and bosons and the ring diagrams have been summed to give an approximate expression for the equation of state valid for the entire temperature range. The near-classical limit (high temperature and low density so that the gas is only slightly degenerate) is discussed in detail, and useful formulae for numerical computation of pressure and internal energy are derived from the general ring approximation to the equation of state. By expanding the chemical potential in powers of the coupling constant e^2, it is possible to eliminate the parametric dependence of pressure and density on the chemical potential and obtain the pressure as a function of density. The classical-limit Debye–Hückel results are obtained when $\hbar$ approaches 0. A general form of the screening length is obtained with the effective screening charge as $z_i e\theta_i$, where θ_i is a measure of the degeneracy of the particle species i.

The contribution to the pressure from the simplest exchange interaction valid for all temperature is evaluated exactly.

The temperature and density range for which the near-classical-limit formulae are valid is discussed.

VOL.
2
961

1. INTRODUCTION

A NUMBER of recent publications have developed general formalisms for dealing with the problems of quantum-statistical mechanics of interacting particles. These approaches involve the application of perturbation methods of quantum field theory to obtain a diagrammatic expansion of the grand partition function of an interacting many-body system. The present paper discusses the application of some of these methods to yield a fundamental and rigorous calculation of the thermodynamic functions (particularly, pressure and internal energy) of a system of point charges interacting only with Coulomb forces. The general theory of a one-component system of point charges obeying Fermi statistics, i.e. an electron gas in a continuous positive-charge background, has been given rather completely by MONTROLL and WARD (1958). In this paper, some general expressions similar to those obtained by MONTROLL and WARD are derived for a multicomponent gas composed of both electrons and positive point charges obeying either Fermi or Bose statistics. The primary aim of this work, however, is to obtain useful analytical results for the equation of state of a fully ionized gas at temperatures and densities for which deviations from the purely classical formulae due to quantum statistics are significant.

* Work performed under the auspices of the U.S. Atomic Energy Commission.

The starting point of any fundamental calculation of the thermodynamic functions of an interacting many-body system at finite temperature is the calculation of the grand canonical partition function, expanded in powers of the coupling constant of the interaction. MATSUBARA has given an elegant field-theoretic treatment of the grand partition function of the electron-phonon system (MATSUBARA, 1955). YANG and LEE (1959) have developed the binary-collision method and applied it to hard-sphere gases. MONTROLL and WARD (1958) have applied the Feynman propagator techniques of field theory to quantum-statistical mechanics and obtained an expansion of the grand partition function in terms of diagrams that are quantum generalizations of the MAYER cluster diagrams. BLOCH and DE DOMINICIS have obtained an expansion in terms of linked diagrams closely related to Rayleigh-Schroedinger perturbation theory (BLOCH and DE DOMINICIS, 1959). More recently, GLASSGOLD, HECKROTTE and WATSON (1959) have obtained the results of BLOCH and DE DOMINICIS by using a theorem due to Hugenholtz. BRITTIN and SAKAKURA (unpublished) have obtained results similar to those of MONTROLL and WARD (1958) by more direct use of field-theory techniques. As a result of the work of these various investigators, the perturbation expansion of the grand partition function may be considered as well understood, and, consequently, this paper is concerned primarily with application of the theory.

The formalism of GLASSGOLD, HECKROTTE and

WATSON will be used, since the fundamental result obtained by them is a convenient starting point for the calculation. They give an expansion of the grand partition function of a one-component system in terms of the sum of all possible linked diagrams:

$$Z_G(\alpha) = \sum_{N=1}^{\infty} Tr\, e^{(\alpha N - \beta H_N)} = Z_0(\alpha)\, e^{\Sigma Q_L}. \qquad (1)$$

Each Q_L is obtained from a graph in which n particles are linked by interactions. Thus, for the Coulomb interaction with coupling constant e^2, such a Q_L is proportional to e^{2r}, where r is the number of interactions; $Z_0(\alpha)$ is the grand partition function of a non-interacting system, but evaluated for $\alpha = \beta\mu$, where μ is the chemical potential per particle of the interacting system.

For application to ionized gases, we consider in this article only a limited set of diagrams out of the multitude of possible diagrams that comprise the expansion indicated by equation (1). This set consists of only the cycle or ring diagrams that are characterized by n particles interacting n times and exchanging momentum q at each interaction. The physical assumption on which this choice is based is that for very-long-range interactions, particularly the Coulomb interaction with its infinite range, the contribution to the sum ΣQ_L in equation (1) from a cluster of n particles interacting cyclically, no matter how far apart, is more important than the contributions from clusters with more than n interactions. The retention of only the ring diagrams thus neglects the ladder diagrams between two particles, which give the all-important contribution for strong short-range forces. At zero temperature and high density, the ring diagrams give the GELL-MANN and BRUECKNER expression for the correlation energy of an electron gas (GELL-MANN and BRUECKNER, 1957) and this result has been shown to be equivalent to the random-phase approximation of BOHM and PINES (1958).

In the high-temperature limit, the ring diagrams give the classical Debye–Hückel equation of state valid for low density (MONTROLL and MAYER, 1941). At any temperature, each ring diagram diverges as $O(-1/q^{2n})$ because of the infinite range of the Coulomb interaction. The sum of all ring diagrams, however, gives a finite result corresponding to a screened Coulomb interaction. The screening length is the same as that obtained from a linearized Thomas-Fermi type of equation, and which reduces to the Debye–Hückel theory in the classical limit.

In Section 2, the GLASSGOLD, HECKROTTE, and WATSON formalism has been extended to show that the Q_L corresponding to particular diagrams can be expressed as iterated temperature integrals similar to the time iterations of the S-matrix in field theory. A temperature-dependent particle propagator is derived. It is then possible to show that this method gives results identical to those of MONTROLL and WARD. For the sum of the ring diagrams, a formal expression for the equation of state is obtained. This formal expression holds for the entire temperature range and gives the GELL-MANN and BRUECKNER theory and the Debye–Hückel theory as limiting cases for zero temperature and very high temperature, respectively. This result is identical with the MONTROLL and WARD expression for a one-component gas, but is obtained in a more compact, and possibly more useful, form.

In Section 3, the complete expression for the sum of the ring diagrams is used to derive limiting results valid for the near-classical limit, for which the gas is slightly degenerate. The near-classical limit is essentially a high-temperature and low density region, however, temperatures and densities are such that quantum statistics cause some deviation from the classical-limit results defined by $\hbar = 0$. Since the grand partition function has been used, the chemical potential appears in the theory as an independent variable in place of the density. Thus, in general, when the grand partition function is used, one obtains parametric equations for pressure and density as functions of $\alpha = \beta\mu$. In the near-classical limit, it is possible to eliminate the implicit dependence on α, and the final expressions obtained for the pressure and energy are functions of density.

In Section 4, the simplest exchange interaction is evaluated, and its contribution to the pressure and energy of an electron gas at all temperatures is obtained. In Section 5 the theory is extended to a multicomponent system of fermions and bosons, and the ring-diagram equation of state of this system is obtained. From this general expression, the thermodynamic functions of a real, fully-ionized gas in the near-classical limit are derived in terms of density as the independent variable.

2. APPLICATION TO A ONE-COMPONENT GAS

In order to thoroughly illustrate and minimize the complexity of the mathematical expressions, a complete discussion of the ring diagrams for a one-component gas will be given first. We consider $\bar{N}$ particles of charge $-e$ and spin s in a volume V and, hence, a number density $\rho = \bar{N}/V$. The total negative charge, $-\bar{N}e$, is balanced by an equal amount of positive charge smeared out continuously through the

VOL
2
196

volume. The Hamiltonian of the system that enters (1) is:

$$H_N = T_N + U_N = -\frac{\hbar^2}{2m}\sum_i^N \nabla_i^2 + \sum_{i<j}^N U(r_{ij}), \quad (2)$$

where $U(r_{ij})$ is the Coulomb potential between point charges at r_i and r_j.

A few relations derived from the grand partition function will be noted here for future reference. The logarithm of the grand partition function gives the pressure of the gas directly,* since we have

$$Z_G(\alpha) \equiv \exp(PV/kT), \quad (3)$$

but Z_G is a function of α. Since PV obtained from (3) is a thermodynamic potential from which the other thermodynamic functions may be derived, it will be given a separate symbol:

$$\Omega(\alpha) \equiv PV. \quad (4)$$

The internal energy is given by:

$$\frac{\bar{E}(\alpha)}{kT} = \frac{1}{Z_G}\sum_N Tr\, H_N \exp(\alpha N - \beta H_N)$$
$$= -\beta\frac{\partial}{\partial\beta}\log Z_G\big|_\alpha = -\beta\frac{\partial}{\partial\beta}\left(\frac{\Omega}{kT}\right)_\alpha, \quad (5)$$

and, similarly, the number of particles $\bar{N}$ is given by:

$$\bar{N} = \frac{\partial}{\partial\alpha}\left(\frac{\Omega}{kT}\right)_\alpha. \quad (6)$$

Since the density $\rho = \bar{N}/V$ is given, equation (6) determines the chemical potential α. Thus, (3), (5), and (6) are a set of parametric equations in α, from which the pressure and internal energy of the gas may be calculated for a given value of density. The potential energy of the interacting gas may be calculated by replacing the coupling constant with $e'^2 = \eta e^2$, so that the pair potential in the Hamiltonian becomes $U_{e'}(r_{ij}) = \eta U(r_{ij})$. One then obtains:

$$\frac{\bar{U}(\alpha)}{kT} = \frac{1}{Z_G}\sum_N Tr\,(e^{\alpha N-\beta T_N}\,\beta U_N e^{-\beta U_N})$$
$$= \lim_{\eta\to 1}\eta\frac{\partial}{\partial\eta}\left(\frac{\Omega(\alpha,\eta)}{kT}\right)_{\beta,\alpha}. \quad (7)$$

In terms of the diagrammatic expansion indicated by (1), the pressure of the gas is given by:

$$\frac{\Omega(\alpha)}{kT} = \log Z_0(\alpha) + Q_1 + \sum_{n=2} Q_n + \sum Q_{\text{non-ring}}. \quad (8)$$

The first term of (8) is the quantum ideal-gas pressure, evaluated for the chemical potential of the interacting system. It should be noted that since α depends on the interaction, even the ideal-gas term depends implicitly on e^2. The quantity Q_1 is the contribution to the pressure from two particles interacting once, and is analogous to the energy of two particles in first-order perturbation theory. As in perturbation theory when applied to interacting fermions or bosons, Q_1 is composed of two parts, a direct interaction and an exchange interaction. The third term is the sum of the ring-diagram contributions. It will be seen that Q_n corresponds to certain terms in the n^{th}-order Rayleigh-Schroedinger perturbation theory. All the non-ring terms indicated in the last term of (8) will be neglected. The non-ring contributions will be discussed in a later article.

The form of $\log Z_0(\alpha)$, the pressure of an ideal quantum gas, is well known, but will be discussed briefly in order to establish notation that will be needed later. The usual form of the quantum ideal-gas grand partition function is for Fermi statistics:

$$Z_0(\alpha_0) = \prod_i (1 + e^{\alpha_0 - \beta p_i^2/2m})^{2s+1}, \quad (9)$$

where p_i are the momenta of free-particle states in a box of volume V, $2s+1$ arises from the spin degeneracy of each state, and α_0 is the chemical potential of the non-interacting gas. Using (9), we obtain:

$$\log Z_0(\alpha)_0 = (2s+1)\sum_i (1 + e^{\alpha_0 - \beta p_i^2/2m})$$
$$= \frac{(2s+1)4\pi V}{(2\pi\hbar)^3}\int_0^\infty dp\, p^2 \log(1 + e^{\alpha_0 - \beta p^2/2m}). \quad (10)$$

If the variable of integration in (10) is changed to $x = p/\bar{p}$, where $\bar{p} = (2mkT)^{1/2}$, and if we integrate once by parts, we have:

$$\log Z_0(\alpha_0) = \bar{N}\frac{(2s+1)4\pi\bar{p}^3}{(2\pi\hbar)^3\rho}$$
$$\cdot\frac{2}{3}\int_0^\infty \frac{dx\, x^4}{1 + \exp(-\alpha_0 + x^2)}. \quad (11)$$

The integral in (11) is an example of a type that occurs repeatedly. Following CHANDRASEKHAR (1957), we shall use the function:*

$$\mathcal{I}_\nu(\alpha_0) = \frac{2}{\Gamma(\nu+1)}\int_0^\infty \frac{dx\, x^{2\nu+1}}{1 + \exp(-\alpha_0 + x^2)}, \quad (12)$$

* For a concise discussion of the grand canonical ensemble see LANDAU and LIFSHITZ (1958).

* Numerical values of the $\mathcal{I}_\nu$ functions may be obtained from the tables of MCDOUGALL J. and STONER E. C. (1938) *Phil. Trans.* **237**, 67. In terms of the F_k functions tabulated by them for $k = 3/2$ and 1/2, the corresponding $\mathcal{I}_\nu$ functions are $\mathcal{I}_{3/2} = (2/3)(2/\sqrt{\pi})F_{3/2}$ and $\mathcal{I}_{1/2} = (2/\sqrt{\pi})F_{1/2}$. They also give values of $F_{-1/2}$ and $F_{-3/2}$, though to less accuracy. These tables have been extended to include the cases for $k = 5/2$, 7/2, 9/2 and 11/2 by BEERM A. C., CHASE M. N. and CHOQUARD P. F. (1955) *Helv. Phys. Acta* **28**, 529.

where the coefficient has been chosen so that the series expansion of (12) begins with $z_0 = e^{\alpha_0}$, the activity of the ideal gas. This series expansion, which converges for $z_0 < 1$, is:

$$\mathscr{I}_\nu(\alpha_0) = z_0 - \frac{z_0^2}{2^{\nu+1}} + \frac{z_0^3}{3^{\nu+1}} - \dots \tag{13}$$

The $\mathscr{I}_\nu$ functions defined in this way have the convenient property, which will be used often, that:

$$\mathscr{I}_\nu'(\alpha) = \mathscr{I}_{\nu-1}(\alpha). \tag{14}$$

It is also convenient to collect the physical quantities that multiply the integral in (11) into a dimensionless parameter ζ, defined as:

$$\zeta = \frac{(2\pi\hbar)^3\rho}{(2s+1)\pi^{3/2}\bar{p}^3}. \tag{15}$$

In terms of the $\mathscr{I}_\nu$ functions and ζ, the ideal Fermi-gas pressure (11) is:

$$\log Z_0(\alpha_0) = \bar{N}\zeta^{-1}\mathscr{I}_{3/2}(\alpha_0). \tag{16}$$

Noting that $\zeta \sim \beta^{-3/2}$ and using (5), we obtain, for the internal energy,

$$\frac{E_0}{kT} = \frac{3}{2}\bar{N}\zeta^{-1}\mathscr{I}_{3/2}(\alpha_0). \tag{17}$$

The chemical potential, $\alpha_0 = \beta\mu_0$, of the non-interacting gas is determined by normalization to the number of particles. If (16) is differentiated with respect to α_0 according to (6) and (14) is used, the defining relation for α_0 is given as:

$$\mathscr{I}_{1/2}(\alpha_0) = \zeta. \tag{18}$$

The ideal gas with Bose–Einstein statistics is treated in the same way beginning with (9), but with the 1 replaced with -1. Corresponding to the $\mathscr{I}_\nu$ functions for Fermi statistics, the corresponding functions for Bose statistics will be denoted by $\mathscr{J}_\nu$ and defined as:

$$\mathscr{J}_\nu(\alpha_0) = \frac{2}{\Gamma(\nu+1)}\int_0^\infty \frac{dx\, x^{2\nu+1}}{e^{-\alpha_0+x^2}-1} = z_0 + \frac{z_0^2}{2^{\nu+1}} + \frac{z_0^3}{3^{\nu+1}} + \dots \tag{19}$$

The $\mathscr{J}_\nu$ functions also have the following useful property:

$$\mathscr{J}_\nu'(\alpha) = \mathscr{J}_{\nu-1}(\alpha). \tag{20}$$

The pressure, energy, and defining relation for α_0 are given by:

$$\log Z_0(\alpha_0) = \bar{N}\zeta^{-1}\mathscr{J}_{3/2}(\alpha_0), \tag{21}$$

$$\frac{E_0}{kT} = \frac{3}{2}\bar{N}\zeta^{-1}\mathscr{J}_{3/2}(\alpha_0), \tag{22}$$

$$\mathscr{J}_{1/2}(\alpha_0) = \zeta. \tag{23}$$

The corresponding results for Boltzmann statistics are obtained by retaining only the first term of either the $\mathscr{I}_\nu$ or $\mathscr{J}_\nu$ functions, i.e. only z_0.

In terms of the preceding results, the ideal-gas term in (8) for the interacting quantum gas, i.e. $e^2 \neq 0$, is:

$$\log Z_0(\alpha) = \begin{cases} \bar{N}\dfrac{\mathscr{I}_{3/2}(\alpha)}{\mathscr{I}_{1/2}(\alpha_0)} & \text{F.D.} \\ \bar{N}\dfrac{\mathscr{J}_{3/2}(\alpha)}{\mathscr{J}_{1/2}(\alpha_0)}, & \text{B.E.} \end{cases} \tag{24}$$

according to the type of statistics. The chemical potential of the real gas may be written as $\alpha = \alpha_0 + \delta\alpha$, where $\delta\alpha$ depends on e^2. At sufficiently high temperature and low density such that $\zeta \ll 1$, (24) reduces to the Boltzmann form:

$$\log Z_0(\alpha) = \bar{N}(z/z_0). \tag{25}$$

The interaction portion of the grand partition function, when expanded as indicated by (1) requires the evaluation of the various Q_L, one for each topologically distinct linked diagram. For a diagram involving n particles initially in states $(\mathbf{p}_1,\sigma_1)$, $(\mathbf{p}_2,\sigma_2)$, $\dots$, $(\mathbf{p}_n,\sigma_n)$ and making r interactions among themselves, the general form of the Q_L for Fermi statistics, as shown by the analysis of GLASSGOLD, HECKROTTE and WATSON, is:

$$Q_L = \sum_{\sigma\sigma'\sigma''\dots} \sum_{pp'p''\dots} f^-(p_1)f^-(p_2)\dots f^-(p_n) \times f^+(p_1')\dots f^+(p_n')f^+(p_1'')\dots \frac{1}{2\pi i}\int_C \frac{dw\, e^{-\beta w}}{w^2} \frac{\langle U\rangle_{FI_{r-1}}\langle U\rangle_{I_{r-1}I_{r-2}}\dots\langle U\rangle_{I_2I_1}\langle U\rangle_{I_1O}}{(w-D_{r-1})(w-D_{r-2})\dots(w-D_1)}, \tag{26}$$

where

$$f^-(p) = \frac{1}{1+\exp(-\alpha+\beta p^2/2m)}$$

$$f^+(p') = 1 - f^-(p') = \frac{1}{1+\exp(+\alpha-\beta p'^2/2m)}. \tag{27}$$

The quantity $f^-(p)$ is the probability that the initial state p is occupied, and $f^+(p')$ the probability that the intermediate state p' is unoccupied. The transition from an initial state O to a final state F takes place through $r-1$ intermediate states denoted by I_1, $I_2, \dots, I_{r-1}$. The summation is over all initial and intermediate spin orientations, and over all allowed values of the initial- and intermediate-state momenta of the n particles. The $D_1, \dots, D_{r-1}$ are the usual energy denominators of Rayleigh–Schroedinger perturbation theory, namely, the difference of the particle energies in the intermediate and initial states. The

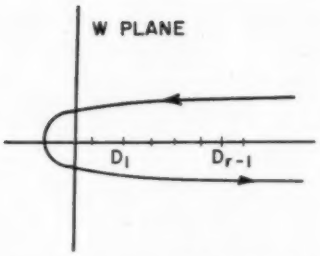


FIG. 1.—Contour of integration C for the w integral of Q_L, (equation (26)).

integral over the complex variable w is taken over the contour shown in Fig. 1 and is chosen so as to enclose the positive real axis, thus giving contributions from the simple poles at $D_1 \ldots D_{r-1}$ and the double pole at $w = 0$. The matrix elements for the pair potential $U(r_{ij})$ have the usual form:

$$\langle p_1'\sigma_1' p_2'\sigma_2' | U | p_1\sigma_1 p_2\sigma_2\rangle = \delta_{\sigma_1\sigma_1'}\delta_{\sigma_1\sigma_1'}\langle p_1' p_2' | U | p_1 p_2\rangle - \delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}\langle p_1' p_2' | U | p_2 p_1\rangle. \quad (28)$$

Both the direct and exchange interaction terms of (28) give rise to the Kronecker $\delta(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_1 - \mathbf{p}_2)$ which expresses momentum conservation. If $\mathbf{p}_1'$ is replaced by $\mathbf{p}_1 + \mathbf{q}$ and $\mathbf{p}_2'$ by $\mathbf{p}_2 - \mathbf{q}$, where $\mathbf{q}$ is the momentum transfer in a momentum-conserving transition, (28) becomes:

$$u(q) - \delta\sigma_1\sigma_2 u(|\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{q}|),$$

with

$$u(q) = \frac{1}{V}\int d\mathbf{r} \exp(i\mathbf{q}.\mathbf{r}/\hbar)U(r). \quad (29)$$

The corresponding Q_L for a cluster of Bose particles is obtained from (21) by replacing $f^-(p)$ and $f^+(p')$ with the analogous quantities for Bose statistics, which are:

$$b^-(p) = \frac{1}{1 + \exp(-\alpha + \beta p^2/2m)}$$

$$b^+(p') = 1 - b^-(p') = \frac{1}{1 - \exp(+\alpha - \beta p'^2/2m)}. \quad (30)$$

The simplest diagram shown in Fig. 2 involves one interaction between two particles with zero momentum transfer, i.e. direct and exchange forward scattering. The contribution to $\beta\Omega$ is:

$$Q_1 = \frac{1}{2!}\sum_{\substack{p_1p_2\\ \sigma_1\sigma_2}} f^-(p_1)f^-(p_2)\frac{1}{2\pi i} \times \int_C \frac{dw\, e^{-\beta w}}{w^2}\langle p_1p_2\sigma_1\sigma_2 | U | p_1p_2\sigma_1\sigma_2\rangle = -\frac{\beta}{2!}(2s+1)^2 \sum_{p_1p_2} f^-(p_1)f^-(p_2) \times \left\{u(0) - \frac{1}{(2s+1)}u(|\mathbf{p}_1 - \mathbf{p}_2|)\right\}. \quad (31)$$

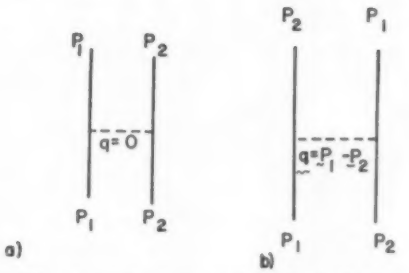


FIG. 2.—Diagrammatic representation of a single interaction between two particles; (a) is the direct interaction, and (b) the exchange interaction. The vertical lines represent the propagation of particles in β-space, the statistical-mechanical analogue of time.

The factor of 1/2! must be present to avoid counting identical states twice. The direct part of (31) represents, essentially, the self-energy of the total charge, $\bar{N}e$, of the particles when spread continuously over the volume V. This self-energy must be cancelled by an equal contribution from the continuous positive background required for electrical neutrality. The second term of (31), which represents forward scattering with exchange, gives a finite non-classical contribution which will be dealt with in Section 4.

The ring diagrams, which are expected to give the principle contribution to the correlation pressure and energy, are shown in Fig. 3 for clusters of 2, 3, and 4 particles. Momentum conservation requires that the momentum transfer be the same for each interaction, and the last interaction returns the cluster of n particles to the states $p_1, \ldots, p_n$ from which they started. In the cases of $n = 2$ and $n = 3$ there is only one distinct diagram, but in general there are $(n-1)!/2$ topologically different diagrams corresponding to the different time orderings of the interactions.* The particle lines shown in each of the diagrams of Fig. 3 may be thought of as lines evenly spaced on the outside of a vertical cylinder. The diagrams used to represent the classical URSELL–MAYER cluster expansion are obtained from this cylindrical representation by looking down the cylinder. The particle lines then become points and are shown as the corners of polygons below the quantum-mechanical diagrams of Fig. 3. The interaction lines in the classical representation encircle the cylinder as a ring. The $3!/2 = 3$ quantum diagrams for $n = 4$ (Fig. 3, c, d

* GELL-MANN and BRUECKNER (1957) use a different diagrammatic representation, and in their Fig. 1 for $n = 4$ show $3! = 6$ diagrams, and, in general, count $(n-1)!/2$ *pairs* of diagrams. Each member of a pair represents one of the two directions around the ring in the MAYER diagrams. The two diagrams of a pair differing only by opposite directions around the ring are not topologically distinct and, of course, give equal contributions.

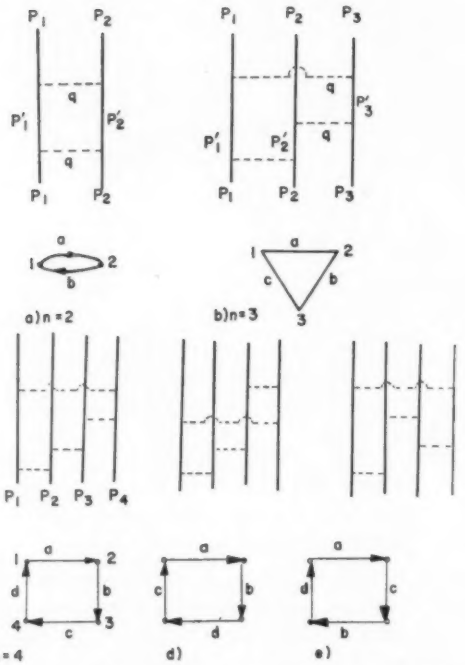


FIG. 3.—The ring diagrams for clusters of 2, 3 and 4 particles. In the quantum-mechanical case the particles propagate along the vertical lines, which are connected by horizontal interaction lines that show the time ordering of the interactions. In the classical MAYER diagrams shown below each quantum diagram, the time order of the interactions is indicated by the letters *a*, *b*, *c*, and *d*.

and e) can be shown in the classical MAYER representation only by writing in the order of the interactions as shown on the diagrams. Only the direct-interaction diagrams are shown in Fig. 3. The contribution of exchange interactions in the ring diagrams will be treated in detail in a later article.

If the general form (26) is used, the one diagram for $n = 2$ gives the contribution:

$$Q_2 = \frac{1}{2}\sum_{\substack{p_1 p_2 q \\ \sigma_1 \sigma_2}} f^-(p_1)f^+(p_1')f^-(p_2)f^+(p_2')u(q)^2 R_2(\beta), \quad (32)$$

where

$$R_2(\beta) = \frac{1}{2\pi i}\int_C \frac{dw\, e^{-\beta w}}{w^2}\cdot\frac{1}{w - D_1},$$

$$D_1 = \varepsilon(p_1') + \varepsilon(p_2') - \varepsilon(p_1) - \varepsilon(p_2)$$

$$= \frac{1}{2m}[(p_1'^2 - p_1^2) + (p_2'^2 - p_2^2)]. \quad (33)$$

Converting the sums to integrations, we have:

$$Q_2 = \frac{(2s+1)^2}{2}\left(\frac{V}{(2\pi\hbar)^3}\right)^3 \int d\mathbf{q} u(q)^2$$

$$\int d\mathbf{p}_1 f^-(p_1)f^+(p_1')\int d\mathbf{p}_2 f^-(p_2)f^+(p_2')R_2(\beta). \quad (34)$$

The contour integral (33) is elementary, and gives:

$$R_2(\beta) = \frac{e^{-\beta D_1}}{D_1^2} - \frac{1}{D_1^2} + \frac{\beta}{D_1}$$

$$= \frac{\beta^2}{2!} - \frac{\beta^3 D}{3!} + \frac{\beta^4 D^2}{4!} - \ldots. \quad (35)$$

The first term of the series expansion of $R_2(\beta)$ gives the classical result for Q_2, valid at high temperature, i.e. β approaches 0.

For $n > 2$ there are $(n-1)!/2$ time orderings of the interactions among the n particles. A particular diagram for a time ordering will be denoted by the subscript α, and the contribution to $\beta\Omega$ from a cluster of n particles will be:

$$Q_n = \sum_{\alpha=1}^{(n-1)!/2} Q_{n\alpha}.$$

The combinatorial factor for each diagram for $n > 2$ is $n!/n! = 1$. The $1/n!$ is required to remove the degeneracy for identical states, i.e. when $\mathbf{p}_1 = \mathbf{p}_2 = \ldots = \mathbf{p}_n$, while the $n!$ is the number of ways the n particles may be placed on the ring. For $n = 2$ however, the factor is 1/2!, since there is only one possible diagram. The $Q_{n\alpha}$ for a given diagram is:

$$Q_{n\alpha} = \sum_{\substack{q p_1 \ldots p_n \\ \sigma_1 \ldots \sigma_1}} f^-(p_1)f^+(p_1')\ldots f^-(p_n)f^+(p_n')u(q)^n R_n(\beta)$$

$$= (2s+1)^n\left(\frac{V}{(2\pi\hbar)^3}\right)^{n+1}\int d\mathbf{q} u(q)^n$$

$$\int d\mathbf{p}_1 f^-(p_1)f^+(p_1')\ldots\int d\mathbf{p}_n f^-(p_n)f^+(p_n')R_n(\beta), \quad (36)$$

where

$$R_n(\beta) = \frac{1}{2\pi i}\int_C \frac{dw\, e^{-\beta w}}{w^2}\prod_i^{n-1}\frac{1}{(w - D_{\sigma i})}. \quad (37)$$

The complex integral may be evaluated conveniently by expanding the product in (37) in partial fractions. The result, as shown in Appendix A, is:

$$R_n(\beta) = \sum_{i=1}^{n-1}\frac{a_i}{D_{\alpha i}^2}(e^{-\beta D\alpha_i} - 1)$$

$$+ (-1)^n\beta\prod_i^{n-1}\frac{1}{D_{\alpha i}}, \quad (38)$$

where

$$a_i = \prod_{j\neq i}^{n-1}\frac{1}{(D_{\alpha i} - D_{\alpha j})},$$

and has the series expansion:

$$R_n(\beta) = (-1)^n\left\{\frac{\beta^n}{n!} - \frac{\beta^{n+1}}{(n+1)!}\sum_i D_{\alpha i}\right.$$

$$\left. + \frac{\beta^{n+2}}{(n+2)!}(\sum D_{\alpha i}^2 + \sum_{i<j} D_{\alpha i}D_{\alpha j}) - \ldots\right\} \quad (39)$$

VOL 2 196

From (38) and (39), the zero-temperature-limit and the classical-limit forms of Q_n may be easily seen. In the very-high-temperature case, which gives the classical limit only the first term of (39) remains. Also, the probability functions $f^-(p)$ and $f^+(p')$ become:

$$f^-(p) \to ze^{-\beta p^2/2m}$$

$$f^+(p') \to 1,$$

so that

$$(2s+1)\frac{V}{(2\pi\hbar)^3}\int d\mathbf{p} f^-(p) = \bar{N}(z/z_0),$$

since (18) reduces to $z_0 = \zeta$. Hence, the classical form for Q_n with the use of (36), becomes:

$$Q_n \text{ classical} = \frac{(-1)^n}{2n}\left(\frac{z}{z_0}\right)^n \beta^n \rho^n \frac{V}{(2\pi\hbar)^3}\int d\mathbf{q}(Vu(q))^n. \tag{40}$$

Equation (40) agrees with the classical form of the ring-diagram contribution obtained by MONTROLL and WARD. This result is similar to the ring-cluster-integral contribution of the MAYER cluster expansion (MONTROLL and MAYER, 1941).* Note that (40) does not actually depend directly on density, since $(z/z_0)\rho \sim z/\lambda_{\text{th}}^3$, where λ_{th} is the thermal de Broglie wavelength defined as $\lambda_{\text{th}} = \hbar/\bar{p}$. It is evident from the expansion (39) that the $Q_{n\alpha}$ are equal to $O(\beta^n)$ for every-time ordering, and that quantum-mechanical effects enter when the $O(\beta^{n+1})$ term must be retained. Since, for the Coulomb potential, $u(q)$ is proportional to $1/q^2$, the leading term of Q_n diverges as $d\mathbf{q}/q^{2n}$ and the Debye–Hückel theory is obtained by formally summing the divergent terms before doing the q integration. The second term of (39) is of order q^2; hence, the retention of higher-order terms in (39), to obtain quantum corrections to the classical result, corresponds to summing lesser divergences.

At zero temperature ($\beta \to \infty$), only the second term of (38) contributes to Q_n. The probability functions $f^-(p)$ and $f^+(p')$ reduce to:

$$f^-(p) = \begin{cases} 1 & p < p_F \\ 0 & p > p_F \end{cases}$$

$$f^+(p) = \begin{cases} 0 & p' < p_F \\ 1 & p' > p_F, \end{cases}$$

where p_F is the Fermi momentum of the interacting system. (The chemical potential as $T \to 0$ is $\alpha = \beta p_F^2/2m$.) The contribution to Ω from a cluster of n particles is:

$$\lim_{\beta\to\infty} Q_n/\beta = (-1)^n(2s+1)^n\left(\frac{V}{(2\pi\hbar)^3}\right)^{n+1} \times \int d\mathbf{q} u(q)^n \int_{\substack{p<p_F \\ |\mathbf{p}+\mathbf{q}|>p_F}} d\mathbf{p}_1 \ldots \times \int d\mathbf{p}_n \sum_{\alpha=1}^{(n-1)!/2} \prod_i^{n-1} \frac{1}{D_{\alpha i}} \tag{41}$$

Equation (41) is the form of Rayleigh–Schroedinger perturbation theory used by GELL-MANN and BRUECKNER in their calculation of the correlation energy of a completely degenerate electron gas.

The Q_L, as defined by (26), is a term of a perturbation expansion of the grand partition function using plane waves as the basis functions, and is valid at every temperature. To obtain a more convenient form for intermediate temperatures, we use an iterated expression for the complex integral $R_n(\beta)$:

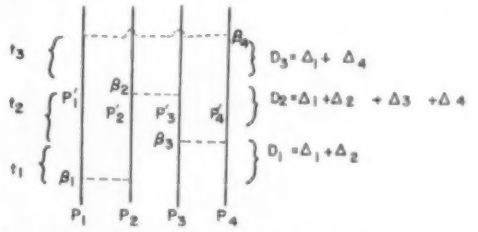


FIG. 4.—One of the diagrams for a cluster of four particles showing the intermediate temperatures corresponding to interaction 'times.' Also shown are the energy denominators for each of the intermediate states.

$$R_{n\alpha}(\beta) = (-1)^n \int_0^\beta d\beta_n \int_0^{\beta_n} dt_{n-1}\, e^{-D_{\alpha n-1}t_{n-1}} \times \int_0^{\beta_n - t_{n-1}} dt_{n-2}\, e^{-D_{\alpha n-2}t_{n-2}} \ldots \times \int_0^{\beta - t_{n-1} - \ldots - t_2} dt_1\, e^{-D_{\alpha 1}t_1}, \tag{42}$$

which may be proved by induction to give (38). Next, we want a change of integration variables that will separate the particle momenta in the $D_{\alpha i}$ so that only one momentum appears in each integral. To effect this separation a set of intermediate temperatures analogous to interaction times are introduced according to the scheme indicated in Fig. 4 for one of the diagrams (Fig. 3(e)) of a cluster of four particles. The temperatures β_1, β_2, β_3, and β_4 indicate the 'times' of the interactions subject to the restriction for the diagram of Fig. 4 that $\beta > \beta_4 > \beta_2 > \beta_3 > \beta_1$.

* For a comparison with the ring-cluster integrals of the MAYER expansion, see Appendix B of the paper by MONTROLL and WARD.

The t variables in (42) are thus the 'time' differences between interactions:

$$t_1 = \beta_3 - \beta_1$$
$$t_2 = \beta_2 - \beta_3$$
$$t_3 = \beta_4 - \beta_2, \tag{43}$$

and indicate the duration of the three intermediate states with energy denominators D_1, D_2 and D_3, respectively. The energy denominators are sums of particle excitation energies:

$$\Delta = \varepsilon(p') - \varepsilon(p) = (p'^2 - p^2)/2m$$
$$= (q^2 + 2\mathbf{q} \cdot \mathbf{p})/2m. \tag{44}$$

In terms of the β variables of (43), $R_{4\alpha}(\beta)$ for this diagram becomes:

$$R_{4\alpha}(\beta) = \int_0^\beta d\beta_4 \int_0^{\beta_4} dt_3\, e^{-(\Delta_1+\Delta_4)t_3} \int_0^{\beta_4 - t_3} dt_2\, e^{-(\Delta_1+\Delta_2+\Delta_3+\Delta_4)t_2} \int_0^{\beta_4 - t_3 - t_2} dt_1\, e^{-(\Delta_1+\Delta_2)t_1}$$
$$= \int_0^\beta d\beta_4 \int_0^{\beta_4} d\beta_2 \int_0^{\beta_2} d\beta_3 \times \int_0^{\beta_3} d\beta_1\, e^{-(\beta_4-\beta_1)\Delta_1 - (\beta_2-\beta_1)\Delta_2 - (\beta_2-\beta_3)\Delta_3 - (\beta_4-\beta_3)\Delta_4}. \tag{45}$$

For any diagram, $R_{n\alpha}(\beta)$ may be written down by inspection in the same fashion. By interchanging the order of the temperature and momentum integrations, we may further write the Q for the diagram shown in Fig. 4 as:

$$Q_{4\alpha} = \frac{V}{(2\pi\hbar)^3} \int d\mathbf{q} u(q)^4 H_{4\alpha}(q,\beta)$$
$$H_{4\alpha}(q,\beta) = \int_0^\beta d\beta_4 \int_0^{\beta_4} d\beta_2 \int_0^{\beta_2} d\beta_3 \int_0^{\beta_3} d\beta_1 G(\beta_4 - \beta_1) \times G(\beta_2 - \beta_1)\, G(\beta_2 - \beta_3) G(\beta_4 - \beta_3), \tag{46}$$

where

$$G(q,\tau) = \frac{(2s+1)V}{(2\pi\hbar)^3} \int d\mathbf{p} f^-(p) f^+(p')\, e^{-\tau\Delta}$$

for $\quad 0 < \tau < \beta$

$$= 0 \quad \text{for} \quad \tau < 0, \quad \tau > \beta. \tag{47}$$

The quantity $G(q,\tau)$ is the temperature-dependent particle propagator after excitation by the momentum transfer q. If the product $f^-(p)f^+(p')$ is expanded in powers of the activity, $z = e^\pi$, one obtains for (47):

$$G(q,\tau) = \frac{(2s+1)V}{(2\pi\hbar)^3} \sum_{r=1} (-1)^{r+1} z^r \sum_{j=0}^{r-1} \int d\mathbf{p} \exp\left[-\frac{p^2}{2m}[(r-j)\beta - \tau]\right] \times \exp\left[-\frac{(\mathbf{p}+\mathbf{q})^2}{2m}(j\beta + \tau)\right], \tag{48}$$

which is identical to the fermion propagator obtained by MONTROLL and WARD (their equation 5.17). When the momentum integration is carried out, (48) becomes:

$$G(q,\tau) = \frac{(2s+1)V}{(2\pi\hbar)^3} \sum_{r=1} (-1)^{r+1} z^r \sum_{j=0}^{r-1} \left(\frac{2\pi m}{\beta r}\right)^{3/2} \times \exp\left[-\frac{q^2}{2m\beta r}(j\beta + \tau)(r\beta - j\beta - \tau)\right]$$
$$= \bar{N}\zeta^{-1} \sum_{r=1} \frac{(-1)^{r+1} z^r}{r^{3/2}} \sum_{j=0}^{r-1} \exp\left[-(q/\bar{p})^2 \times (j + \tau/\beta)(r - j - \tau/\beta)/r\right]. \tag{49}$$

If (49) is used, a useful symmetry property of the propagator may be proved. One notes that $G(\tau)$ is equal to $\Sigma_r G_r(\tau)$ and that $G_r(\tau)$ has the form:

$$G_r(\tau) = \sum_{j=0}^{r-1} f[(j + \tau/\beta)(r - j - \tau/\beta)]. \tag{50}$$

If we change the summation variable from j to $j' = r - j - 1$, then (50) becomes:

$$G_r(\tau) = \sum_{j'=0}^{r-1} f[(r - j' - 1 + \tau/\beta) \times (j' + 1 - \tau/\beta)] = G_r(\beta - \tau), \tag{51}$$

and consequently:

$$G(q, \beta - \tau) = G(q,\tau). \tag{52}$$

The $Q_{n\alpha}$ for any time ordering may be written in the following form, similar to (46):

$$H_{n\alpha}(\beta) = \int_0^\beta \ldots \int_0^{} d\beta_n \ldots d\beta_1 G(\beta_n - \beta_1) \times G(\pm(\beta_2 - \beta_1) \ldots G(\pm(\beta_n - \beta_{n-1})). \tag{53}$$

A particular time ordering is expressed by choosing the upper limits of the integrations in (53) as a permutation of the intermediate temperatures, $\beta_n \ldots \beta_2$.

For example, with $n = 4$, the contributions of the three diagrams of Fig, 3 are obtained by choosing the limits of the temperature integrations, respectively, so

that $\beta > \beta_4 > \beta_3 > \beta_2 > \beta_1$, $\beta > \beta_3 > \beta_4 > \beta_2 > \beta_1$, and $\beta > \beta_4 > \beta_2 > \beta_3 > \beta_1$. Finally, the sum of the $(n-1)!/2$ diagrams for an n-particle cluster may be conveniently written as a single iterated integral:

$$Q_n = \frac{V}{(2\pi\hbar)^3}\int d\mathbf{q} u(q)^n H_n(q,\beta)$$

$$H_n(q,\beta) = \sum_{\alpha=1}^{(n-1)!/2} H_{n\alpha}$$
$$= \frac{(-1)^n}{2n}\int_0^\beta \cdots \int_0^\beta d\beta_1 \ldots d\beta_n G(|\beta_n - \beta_1|)$$
$$\times G(|\beta_2 - \beta_1|) \ldots G(|\beta_n - \beta_{n-1}|). \quad (54)$$

The iterated temperature integral of (54) may be reduced to a sum of the n^{th} power of the eigenvalues of a homogeneous Fredholm integral equation with the propagator as a kernel:

$$\lambda_k(q)\psi_k(\beta') = \int_0^\beta d\beta'' \psi_k(\beta'')G(q, |\beta' - \beta''|). \quad (55)$$

The solutions of (55), $\psi_k(\beta')$, form a complete, normalized, orthogonal set of functions corresponding to the eigenvalues $\lambda_k(q)$. Using Mercer's theorem, we may expand the propagator as (TRICOMI, 1957):

$$G(q, |\beta' - \beta''|) = \sum_{k=-\infty}^{\infty} \lambda_k(q)\psi_k^*(\beta')\psi_k(\beta''). \quad (56)$$

If (56) and the orthonormality of $\psi_k(\beta')$ are used, the iterated integral of (54) becomes:

$$H_n(q,\beta) = \frac{(-1)^n}{2n}\sum_{k=-\infty}^{\infty} \lambda_k{}^n(q). \quad (57)$$

The $\lambda_k(q)$ are obtained from the integral equation (55). The procedure is to use an assumed form of the eigenfunctions, namely:

$$\psi_k(\beta') = \beta^{-1/2} e^{2\pi i k\beta'/\beta}, \quad (58)$$

and verify that this choice satisfies the integral equation. The right-hand side of (55) may be written in two parts as:

$$\int_0^{\beta'} d\beta'' \psi_k(\beta')G(\beta' - \beta'') + \int_{\beta'}^{\beta} d\beta'' \psi_k(\beta'')G(\beta'' - \beta').$$

By changing variables to $\tau = \beta' - \beta''$ in the first part and $\beta - \tau = \beta' - \beta''$ in the second part, and using the assumed sinusoidal form of $\psi_k(\beta')$, one verifies that (58) does satisfy the integral equation and one is left with the eigenvalue as:

$$\lambda_k(q) = \int_0^{\beta'} d\tau\, e^{-2\pi i k\tau/\beta} G(\tau)$$
$$+ \int_{\beta'}^{\beta} d\tau\, e^{-2\pi i k\tau/\beta} G(\beta - \tau). \quad (59)$$

Using the symmetry property of the propagator, equation (52), we find that the eigenvalue reduces to a single integral:

$$\lambda_k(q) = \int_0^\beta d\tau\, e^{-2\pi i k\tau/\beta} G(q,\tau), \quad (60)$$

which agrees with the MONTROLL and WARD result, but has been derived differently.* The eigenvalue for bosons is also given by (60), but the boson propagator is:

$$G(q,\tau) = \frac{(2s+1)V}{(2\pi\hbar)^3}\int d\mathbf{p} b^-(p) b^+(p')\, e^{-\tau\Delta}, \quad (61)$$

Equation (61) has a power-series expansion in z given by the fermion propagator expansion (49), but with the $(-1)^{r+1}$ replaced by 1.

Changing the variable of the temperature integration in (60) to $x = \tau/\beta$, and the momentum integration in the propagator to $y = p/\bar{p}$, and multiplying and dividing by ζ, we may rewrite the eigenvalue expression as:

$$\lambda_k(q) = V\rho\beta\mathscr{L}(q|\bar{p}, 2\pi k), \quad (62)$$

where

$$\mathscr{L}(q|\bar{p}, 2\pi k) = \int \frac{d\mathbf{y}}{\pi^{3/2}} \frac{f^-(y)f^+(y')}{\zeta}$$
$$\times \int_0^1 dx\, e^{-x(y'^2 - y^2) + 2\pi i k x}. \quad (63)$$

If the expansion of the propagator (49) is used, an expansion of the dimensionless function $\mathscr{L}(q|\bar{p}, 2\pi k)$ in powers of $(q|\bar{p})^2$ may be obtained, convergent for all values of q. For large values of the momentum transfer, the asymptotic form of $\mathscr{L}(q|\bar{p}, 2\pi k)$ is $O(\bar{p}^2/q^2)$ for all k. For small q, the $k = 0$ eigenvalue behaves as $\mathscr{L}(q|\bar{p}, O) \propto$ constant $- O(q^2)$, whereas for $k \neq 0$, it begins with $O(q^2)$. It should be noted that the eigenvalue as given by (62) is still a function of the activity z, not the density ρ, since ρ/ζ is independent of density. The misleading way of writing λ_k in (62) is, however, convenient for later work.

Returning now to the ring-integral contribution to $\beta\Omega$, we find:

$$(\beta\Omega)_{\text{ring}} = \sum_{n=2}^{\infty} Q_n$$
$$= \frac{V}{(2\pi\hbar)^3}\sum_{n=2}\frac{(-1)^n}{2n}\int d\mathbf{q} \sum_{k=-\infty}^{\infty} (u(q)\lambda_k(q))^n. \quad (64)$$

* In the present derivation it was not necessary to assume a periodic extension of the propagator, i.e. $G(\tau + \beta) = G(\tau)$, as they did. The proof of the symmetry relation, equation (52), was shown to the author by HOWARD LEVINE. It was also pointed out by Dr. LEVINE that the propagator is a function of $|\beta' - \beta''|$ only when both β' and β'' are in the range 0 to β and, since the propagator is zero for β' or β'' outside of this range, the same is true for $\psi_k(\beta')$. Thus, $\psi_k\beta')$ is a sinusoidal function with only one period; it is given by (58) for $0 < \beta' < \beta$ but is otherwise zero.

Equation (64) is well-defined for any pair potential $U(r)$ that has a Fourier transform; however, the ring diagrams are a sensible choice for the physical problem only when $U(r)$ has an extremely long range. The Coulomb potential has a Fourier transform when defined as $U(r) = (e^2/r)e^{-\varepsilon r}$, with the result using (29).

$$u(q) = \frac{4\pi e^2\hbar^2}{V(q^2 + \varepsilon^2)}. \tag{65}$$

If the limit $\varepsilon \to 0$ is taken, then each Q_n in (64) diverges. However, by retaining $\varepsilon \neq 0$, Q_n is formally finite and the interchange of the q integration and n summation may be justified. The n summation is convergent only for q and ε large enough so that $u(q)\lambda_k(q) < 1$, but nevertheless represents the integrand correctly for all q. After interchanging the order of the k and n summations, we find that (64) becomes:

$$(\beta\Omega)_{\text{ring}} = \frac{1}{2}\cdot\frac{V4\pi}{(2\pi\hbar)^3}\int_0^\infty dq\, q^2 \times \sum_k \{u(q)\lambda_k(q) - \ln(1 + u(q)\lambda_k(q))\}. \tag{66}$$

For the Coulomb potential, using (65) and (62), we have:

$$u(q)\lambda_k(q) = \frac{(4\pi e^2\beta\rho)\hbar^2\mathscr{L}(q/\bar{p}, 2\pi k)}{q^2 + \varepsilon^2}. \tag{67}$$

After taking the limit $\varepsilon \to 0$, we find that the integrand of (66) is still finite at $q = 0$. It is convenient to rewrite (67) in terms of the dimensionless parameter γ:

$$\gamma^2 = \frac{(4\pi e^2\beta\rho)\hbar^2}{\bar{p}^2} = (\lambda_{\text{th}}/\lambda_D)^2 \quad , \tag{68}$$

where $\lambda_D = (4\pi e^2\beta\rho)^{-1/2}$ is the classical Debye screening length for a one-component gas. After changing the integration variable in (66) to $v = q/\bar{p}$, and adding the ideal gas term, we find the ring-integral approximation to the equation state of a one-component gas to be:

$$\frac{\Omega(\alpha)}{kT} = \bar{N}\frac{\mathscr{I}_{3/2}(\alpha)}{\zeta} + \frac{1}{2}\cdot\frac{4\pi V\bar{p}^3}{(2\pi\hbar)^3} \times \int_0^\infty dv\, v^2 \sum_k \left\{(\gamma^2/v^2)\,\mathscr{L}(v, 2\pi k) - \ln(1 + (\gamma^2/v^2)\mathscr{L}(v, 2\pi k))\right\}. \tag{69}$$

The chemical potential α is determined by normalization to the number of particles $\bar{N}$ in the volume V; when (69) is differentiated with respect to α according to (6) and $\bar{N}$ is cancelled, the following remains as the defining equation for α:

$$1 = \frac{\mathscr{I}_{1/2}(\alpha)}{\zeta} + \frac{1}{2}\frac{4\pi\rho\bar{p}^3}{(2\pi\hbar)^3} \times \int_0^\infty dv \sum_k \frac{\gamma^4\mathscr{L}(v, 2\pi k)}{[v^2 + \gamma^2\mathscr{L}(v, 2\pi k)]}\frac{\partial\mathscr{L}}{\partial\alpha}. \tag{70}$$

In the limit of no interaction, $e^2 = 0$, (70) reduces to (18).

At any temperature, (69) and (70) are parametric equations in α for the equation of state of a one-component gas of fermions or bosons, for densities such that the non-ring and exchange diagrams may be neglected. In the zero-temperature limit, MONTROLL and WARD have shown that the GELL-MANN and BRUECKNER expression of the correlation energy of an electron gas of high density may be derived from (69) and (70). Similarly, in the opposite limit of very high temperature and low density, they obtained from (69) and (70) the classical-limit result of Debye and Hückel. For temperatures and densities between these two extreme limits, the mathematical analysis required for the v integration and k summation in (69) is very much more difficult. The evaluation of (69) for a very degenerate gas at finite temperature and the criterion for the validity of the ring-diagram approximation will be deferred to a later article.

3. THE NEAR-CLASSICAL LIMIT

The general expression for the sum of the ring diagrams (66) may be used to obtain useful analytical formulae for a slightly degenerate gas for which the density is high enough so that there are some corrections due to quantum statistics. The true classical limit is obtained by setting $\hbar = 0$ in the near-classical-limit formulae.

The significant contribution to the q integration of (66) comes from $q \sim 0$, and corresponds to interactions between particles at great distances. The classical limit and near-classical-limit form of (66) may be obtained by neglecting terms of $O(q^2)$. Since only the $k = 0$ eigenvalue has a non-zero value at $q = 0$, this approximation simplifies the k summation to one term. To find the $q = 0$ value of λ_0, we note that the $f^-(y)f^+(y')$ factor in the eigenvalue expression (63) may be written as:

$$f^-(y)f^+(y') = f^-(y)(1 - f(y)) - f^-(y)(f^-(y') - f^-(y)) = \frac{d}{d\alpha}f^-(y) + O(q^2/\bar{p}^2),$$

and, consequently,

$$\mathscr{L}(q|\bar{p}, O) = \frac{1}{\zeta}\frac{d}{d\alpha}\mathscr{I}_{1/2}(\alpha) - O(q^2|\bar{p}^2)$$
$$= \zeta^{-1}\mathscr{I}_{-1/2}(\alpha) - \ldots \quad (71)$$

Since the quantity $\zeta^{-1}\mathscr{I}_{-1/2}(\alpha)$ occurs fairly often and always in connexion with e^2, it is convenient to denote it by a separate symbol:

$$\theta^2(\alpha) \equiv \zeta^{-1}\mathscr{I}_{-1/2}(\alpha). \quad (72)$$

Using (71) and dropping quantities of $O(q^2)$, we have, for the ring-integral contribution to $\beta\Omega$,

$$(\beta\Omega)_{\text{ring}} \cong \frac{1}{2}\cdot\frac{4\pi V}{(2\pi\hbar)^3}\int_0^\infty dqq^2\{x_0(q) - \ln(1+x_0(q))\}, \quad (73)$$

with

$$x_0(q) = u(q)\lambda_0(q) \cong (\hbar/\lambda_D q)^2\theta^2(\alpha).$$

The q integral is elementary in this approximation and gives:

$$(\beta\Omega)_{\text{ring}} \cong \frac{2\pi V}{(2\pi\hbar)^3}\cdot\frac{\pi}{3}\left(\frac{\hbar\theta(\alpha)}{\lambda_D}\right)^3$$
$$= \frac{2}{3}\bar{N}\Lambda\theta^3(\alpha) = \frac{V\theta^3(\alpha)}{12\pi\lambda_D{}^3}, \quad (74)$$

where

$$\Lambda = \frac{e^2/2\lambda_D}{kT} = \sqrt{\pi}\, e^3\beta^{3/2}\rho^{1/2}. \quad (75)$$

The dimensionless parameter Λ is the ratio of the classical Debye–Hückel potential energy per electron, $e^2/2\lambda_D$, to the thermal energy kT. In the near-classical limit it may be expected that the ring diagrams give the leading contribution to the thermodynamic functions of a gas when Λ is small.

If the ideal gas term and (74) are retained, the logarithm of the grand partition function is given by:

$$\beta\Omega(\alpha) = \bar{N}\left\{\zeta^{-1}\mathscr{I}_{3/2}(\alpha) + \frac{2}{3}\Lambda\theta^3(\alpha)\right\}. \quad (76)$$

Noting that $\Lambda\zeta^{-3/2}$ has a temperature dependence of $\beta^{-3/4}$, we find for the internal energy:

$$\frac{E(\alpha)}{kT} = -\beta\frac{\partial}{\partial\beta}\left(\frac{\Omega}{kT}\right)_\alpha$$
$$= \bar{N}\left\{\frac{3}{2}\zeta^{-1}\mathscr{I}_{3/2}(\alpha) + \frac{1}{2}\Lambda\theta^3(\alpha)\right\}. \quad (77)$$

Similarly, the potential energy due to the correlation of the interacting particles is found by using (7), thus:

$$\beta U(\alpha) = -\bar{N}\Lambda\theta^3(\alpha). \quad (78)$$

The chemical potential is determined by:

$$\bar{N} = \frac{\partial}{\partial\alpha}\left(\frac{\Omega}{kT}\right)_\beta = \bar{N}\left\{\frac{\mathscr{I}_{1/2}(\alpha)}{\zeta}\right.$$
$$\left. + \Lambda\left(\frac{\mathscr{I}_{-1/2}(\alpha)}{\zeta}\right)^{1/2}\left(\frac{\mathscr{I}_{-3/2}(\alpha)}{\zeta}\right)\right\}. \quad (79)$$

The previous four equations give the thermodynamic functions of a one-component gas in the near-classical limit. For convenience in numerical calculations, it would be desirable if the dependence on α could be eliminated and the pressure and energy written as functions of density. To accomplish this end, we may treat α as a sum of two parts, $\alpha = \alpha_0 + \delta\alpha$, where α_0 is the chemical potential of the non-interacting gas defined by (18), and $\delta\alpha$ is the modification due to the interaction. To find $\delta\alpha$, at least to the lowest order of the coupling constant, we expand the $\mathscr{I}_\nu$ functions (or the $\mathscr{J}_\nu$ functions, for Bose statistics) in a Taylor series in $\delta\alpha$ about α_0. Equation (79) becomes:

$$1 = \zeta^{-1}[\mathscr{I}_{1/2}(\alpha_0) + \delta\alpha\mathscr{I}_{-1/2}(\alpha_1) + \ldots]$$
$$+ \Lambda\left(\frac{\mathscr{I}_{-1/2}(\alpha_0+\delta\alpha)}{\zeta}\right)^{1/2}\left(\frac{\mathscr{I}_{-3/2}(\alpha_0+\delta\alpha)}{\zeta}\right) \quad (80)$$

Solving for $\delta\alpha$ from (80), we have:

$$\delta\alpha = -\Lambda\theta_0\frac{\mathscr{I}_{-3/2}(\alpha_0)}{\mathscr{I}_{-1/2}(\alpha_0)} + O(\Lambda^2), \quad (81)$$

where

$$\theta_0{}^2 \equiv \theta^2(\alpha_0) = \frac{\mathscr{I}_{-1/2}(\alpha_0)}{\mathscr{I}_{1/2}(\alpha_0)} = 1 - \frac{z_0}{2^{3/2}}$$
$$+ \left(\frac{2}{3^{3/2}} + \frac{1}{8}\right)z_0{}^2 - \ldots \quad (82)$$

The quantity $\theta_0{}^2$ is a measure of the degeneracy of a Fermi gas. In the high-temperature limit $\theta_0 \to 1$, and in the zero-temperature limit $\theta_0 \to 0$. Similarly, for a Bose gas the degeneracy parameter is:

$$\theta_0{}^2 = \frac{\mathscr{I}_{-1/2}(\alpha_0)}{\mathscr{I}_{1/2}(\alpha_0)}$$
$$= 1 + \frac{z_0}{2^{3/2}} + \left(\frac{2}{3^{3/2}} + \frac{1}{8}\right)z_0{}^2 + \ldots \quad (83)$$

and is greater than 1 for any deviation from classical statistics. The quantity $\delta\alpha$ may be conveniently written as:

$$\delta\alpha = -\Lambda\theta_0{}^3 + \Lambda\theta_0\eta, \quad (84)$$

where

$$\eta = \frac{\mathscr{I}_{-1/2}(\alpha_0)}{\mathscr{I}_{1/2}(\alpha_0)} - \frac{\mathscr{I}_{-3/2}(\alpha_0)}{\mathscr{I}_{-1/2}(\alpha_0)}$$
$$= \frac{z_0}{2^{3/2}} - \left(\frac{4}{3^{3/2}} + \frac{3}{8}\right)z_0{}^2 + \ldots \quad (85)$$

VOL. 2 1961

for Fermi statistics, and

$$\eta = \frac{\mathscr{J}_{-1/2}(\alpha_0)}{\mathscr{J}_{1/2}(\alpha_0)} - \frac{\mathscr{J}_{-3/2}(\alpha_0)}{\mathscr{J}_{-1/2}(\alpha_0)}$$
$$= -\frac{z_0}{2^{3/2}} - \left(\frac{4}{3^{3/2}} + \frac{3}{8}\right) z_0^2 - \ldots \quad (86)$$

for Bose statistics. In the near-classical limit η is small, since $z_0 \sim \zeta \ll 1$.

After expanding $\mathscr{J}_{3/2}(\alpha_0 + \delta\alpha)$ and using (84) in (76) and (77), we have, for the pressure and energy of a one-component gas in terms of *density* through α_0 to order e^3 in the coupling constant,

$$\frac{PV}{kT} = \bar{N}\left\{\zeta^{-1}\mathscr{J}_{3/2}(\alpha_0) - \frac{1}{3}\Lambda\theta_0^3 + \Lambda\theta_0\eta + O(\Lambda^2)\right\}$$
$$\frac{\bar{E}}{kT} = \bar{N}\left\{\frac{3}{2}\zeta^{-1}\mathscr{J}_{3/2}(\alpha_0) - \Lambda\theta_0^3 + \frac{3}{2}\Lambda\theta_0\eta + O(\Lambda^2)\right\}$$
$$\frac{\bar{U}}{kT} = -\bar{N}\Lambda\theta_0^3 + O(\Lambda^2). \quad (87)$$

There is no point in calculating the $O(\Lambda^2)$ terms until some of the non-ring terms that may also give $O(\Lambda^2)$ are evaluated. These expressions give the thermodynamic functions for temperatures and densities for which deviations from the purely classical expressions ($\theta_0^2 = 1$ and $\eta = 0$), using Boltzmann statistics, first begin to appear. They are valid for values of θ_0^2 between 1 and some lower limit of the order of 0·9. For a more degenerate gas (i.e. $\theta_0^2 < 0{\cdot}9$), the approximation of neglecting $O(q^2)$ terms in the evaluation (69) is no longer valid. The complete evaluation of (69) will give (74), plus non-classical terms involving higher powers of γ^2.

The pressure and energy formulae (87) satisfy the virial theorem, $PV = (2/3)\bar{T} + (1/3)\bar{U}$, where $\bar{E} = \bar{T} + \bar{U}$. The average kinetic energy is given by:

$$\bar{T}/kT = \bar{N}\left\{\frac{3}{2}\zeta^{-1}\mathscr{J}_{3/2}(\alpha_0) + \frac{3}{2}\Lambda\theta_0\eta\right\}. \quad (88)$$

The η term is seen to be a non-classical correction to the ideal-gas form of $\bar{T}$, due to the change of probability $f^-(p)$ from

$$\{1 + \exp[-\alpha_0 + \beta p^2/2m]\}^{-1}$$
to
$$\{1 + \exp[-(\alpha_0 + \delta\alpha) + \beta p^2/2m]\}^{-1}$$

as the interaction between particles is turned on.

4. EXCHANGE PRESSURE AND ENERGY

We consider now the contribution to the pressure from the exchange interaction in the simplest diagram, Q_1, i.e. forward scattering with exchange as shown in Fig. 2(a). From (31) of Section 2, this contribution is given by:

$$Q_{1x} = \frac{1}{2}\beta(2s+1)\sum_{p_1 p_2} f^-(p_1) f^-(p_2) u(|\mathbf{p}_1 - \mathbf{p}_2|). \quad (89)$$

After converting the summations to integrations and using the Fourier transform of the Coulomb potential we have:

$$Q_{1x} = \frac{1}{2}\frac{\beta}{(2s+1)}\left(\frac{(2s+1)V}{(2\pi\hbar)^3}\right)^2 \bar{p}^6 \frac{4\pi e^2\hbar^2}{V\bar{p}^2} \times \int\int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{\bar{p}^6} f^-(p_1) f^-(p_2) \frac{\bar{p}^2}{(\mathbf{p}_1 - \mathbf{p}_2)^2}$$
$$= \frac{1}{2}\cdot\frac{\bar{N}}{(2s+1)}\gamma^2\zeta^{-2}\mathscr{J}_x(\alpha), \quad (90)$$

where

$$\mathscr{J}_x(\alpha) = \int\int \frac{d\mathbf{y}_1 d\mathbf{y}_2}{\pi^3}\cdot\frac{f^-(y_1) f^-(y_2)}{(\mathbf{y}_1 - \mathbf{y}_2)^2}. \quad (91)$$

The steps of performing the integration of (91) are indicated in Appendix B. The result is:

$$\mathscr{J}_x(\alpha) = 2\sum_{\substack{r_1=1\\ r_2=1}} \frac{(-1)^{r_1+r_2} e^{(r_1+r_2)\alpha}}{\sqrt{r_1 r_2}\,(r_1 + r_2)}$$
$$= 2\int_{-\infty}^{\alpha} d\alpha' \mathscr{J}_{-1/2}(\alpha')^2. \quad (92)$$

Considering only the ideal-gas term and the direct exchange interaction, we have for $\beta\Omega$

$$\beta\Omega(\alpha) = \bar{N}\{\zeta^{-1}\mathscr{J}_{3/2}(\alpha) + Q_{1x}(\alpha) + \ldots\}. \quad (93)$$

and noting that the temperature dependence of γ^2/ζ_2 is β^{-1}, we find that the internal energy is given by:

$$\beta\bar{E}(\alpha) = \bar{N}\left\{\frac{3}{2}\zeta^{-1}\mathscr{J}_{3/2}(\alpha) + Q_{1x}(\alpha) + \ldots\right\}. \quad (94)$$

Also, since Q_{1x} is $O(e^2)$ in the coupling constant, (7) gives the potential energy due to the exchange interaction as:

$$\beta\bar{U}_x(\alpha) = -\bar{N}Q_{1x}. \quad (95)$$

These results, equations (93), (94), and (95), are all expressed in terms of α which also depends on the magnitude of the exchange interaction, and is determined by:

$$\bar{N} = \frac{\partial}{\partial\alpha}\left(\frac{\Omega}{kT}\right)_\beta = \bar{N}\zeta^{-1}\mathscr{J}_{1/2}(\alpha) + \frac{\partial Q_{1x}}{\partial\alpha} + \ldots \quad (96)$$

The chemical potential is composed of several parts:

$$\alpha = \alpha_0 + (\delta\alpha)_x + (\delta\alpha)_{\text{ring}} + (\delta\alpha)_{\text{non-ring}},$$

where $(\delta\alpha)_{\text{ring}}$ is given by (81) and is $O(e^3)$ in the

coupling constant. The non-ring portion is of higher order in e^2. The exchange portion, $(\delta\alpha)_x$, is $O(e^2)$ and hence may be treated independent of the ring contribution, at least to $O(e^2)$. Consequently, if $\mathscr{I}_{1/2}(\alpha_0 + (\delta\alpha)_x)$ is expanded in Taylor series and the ring contribution is ignored, equation (96) gives:

$$(\delta\alpha)_x = -\frac{1}{\bar{N}\theta_0^2}\frac{\partial Q_{1x}}{\partial\alpha} + \cdots = -\frac{\gamma^2\theta_0^2}{(2s+1)} + O(\gamma^4). \tag{97}$$

By expanding $\mathscr{I}_{3/2}(\alpha_0 + (\delta\alpha)_x)$ and using (97), we may write the pressure and energy as functions of α_0 to $O(e^2)$; the results are:

$$\frac{PV}{kT} = \bar{N}\left\{\zeta^{-1}\mathscr{I}_{3/2}(\alpha_0) - \frac{\gamma^2\theta_0^2}{2(2s+1)}(1-\eta_x(\alpha_0)) + O(\gamma^4)\right\}$$

$$\frac{\bar{E}}{kT} = \bar{N}\left\{\frac{3}{2}\zeta^{-1}\mathscr{I}_{3/2}(\alpha_0) - \frac{\gamma^2\theta_0^2}{(2s+1)}\left(1-\frac{1}{2}\eta_x(\alpha_0)\right) + O(\gamma^4)\right\}, \tag{98}$$

where

$$\eta_x(\alpha_0) = \frac{2\int_{-\infty}^{\alpha_0} d\alpha'\,\mathscr{I}^2_{-1/2}(\alpha')}{\mathscr{I}_{1/2}(\alpha_0)\mathscr{I}_{-1/2}(\alpha_0)} - 1 = \frac{z_0}{6\sqrt{2}} - \left(\frac{1}{3\sqrt{3}} - \frac{1}{8}\right)z_0^2 + \cdots \tag{99}$$

For the kinetic and potential energies separately, we find to $O(e^2)$:

$$\frac{\bar{U}_x}{kT} = -\frac{\bar{N}\gamma^2\theta_0^2}{2(2s+1)}(1+\eta_x(\alpha_0))$$

$$\frac{\bar{T}}{kT} = \bar{N}\left\{\frac{3}{2}\zeta^{-1}\mathscr{I}_{3/2}(\alpha_0) - \frac{\gamma^2\theta_0^2}{(2s+1)}(1-2\eta_x(\alpha_0))\right\} \tag{100}$$

As given by (100), $\bar{U}_x$ and $\bar{T}$ satisfy the virial theorem, $PV = \frac{2}{3}\bar{T} + \frac{1}{3}\bar{U}_x$. In the near-classical limit, the main contribution to PV and $\bar{E}$ from the direct exchange interaction is of order of $\gamma^2\theta_0^2$, a small quantity when compared with the ideal-gas kinetic energy. The $\eta_x(\alpha_0)$ terms are small corrections to an already small quantity, since $z_0 \ll 1$ in the near-classical limit. Consequently, in this limit the increments to $\bar{U}$ and $\bar{T}$ due to the exchange interaction are nearly equal.

The formulae of (98) are valid for the entire temperature range. In the zero-temperature limit the internal-energy expression reduces to the well-known result for the exchange energy of an electron gas given by MÄCKE (1950). To obtain the low-temperature limit, we note that the chemical potential at $T = 0$ is $\mu_0 = p_{0F}^2/2m$, where p_{0F} is the usual Fermi momentum; hence near $T = 0$, we have $\alpha_0 = \beta\mu_0 \cong (p_{0F}/\bar{p})^2$. The asymptotic forms of the various $\mathscr{I}_\nu$ functions are:

$$\mathscr{I}_{1/2}(\alpha_0) = \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{x^2\,dx}{1+e^{-\alpha_0+x^2}}$$

$$\xrightarrow[T\sim 0]{} \frac{2}{\sqrt{\pi}}\int_0^{\alpha_0^{1/2}} x^2\,dx = \frac{2\alpha_0^{3/2}}{3\sqrt{\pi}}$$

$$\mathscr{I}_{3/2}(\alpha_0) = \frac{4\alpha_0^{5/2}}{15\sqrt{\pi}}, \qquad \mathscr{I}_{-1/2}(\alpha_0) = \frac{\alpha_0^{1/2}}{\sqrt{\pi}}. \tag{01}$$

Hence, near $T = 0$,

$$\theta_0^2 \cong \frac{3}{2\alpha_0} = \frac{3}{2}\left(\frac{\bar{p}}{p_{0F}}\right)^2. \tag{102}$$

Using the definition of $\eta_x(\alpha_0)$ and (101), we find:

$$\lim_{T=0}\eta_x(\alpha_0) = \frac{1}{2}. \tag{103}$$

Also, using the definitions of γ^2 and ζ and (101), we find near $T = 0$:

$$\frac{\gamma^2\theta_0^2}{2s+1} = \frac{1}{kT}\frac{e^2p_{0F}}{\pi\hbar}, \qquad \zeta = \frac{4}{3\sqrt{\pi}}\left(\frac{p_{0F}}{\bar{p}}\right)^2. \tag{104}$$

When these various results are used in (98), the zero-temperature limit of the pressure and energy of a Fermi gas is given by the following:

$$PV = \bar{N}\left\{\frac{2}{5}\left(\frac{p_{0F}^2}{2m}\right) - \frac{e^2p_{0F}}{4\pi\hbar} + \cdots\right\},$$

$$\bar{E} = \bar{N}\left\{\frac{3}{5}\left(\frac{p_{0F}^2}{2m}\right) - \frac{3e^2p_{0F}}{4\pi\hbar} + \cdots\right\}. \tag{105}$$

The second term of the energy expression is the zero-temperature exchange energy derived by MÄCKE from first-order perturbation theory. The exchange energy and pressure have a density dependence of $\rho^{-1/3}$ in the $T = 0$ limit, and of ρ in the high-temperature limit. Between these two limits there is no simple dependence on density.

For each direct interaction in the ring diagrams, an exchange interaction is also possible and will give rise to exchange correlation energy and pressure. For a cluster of two particles interacting twice, once directly and once with exchange, the contribution to the pressure and energy is finite at all temperatures,

since the term $1/q^2$ for a direct interaction is replaced by $1/(\mathbf{q}+\mathbf{p}_1-\mathbf{p}_2)^2$ for an exchange interaction. This contribution was calculated numerically by GELL-MANN and BRUECKNER for the zero-temperature electron gas (their $\varepsilon_b^{(2)}$) and found to contribute only in the constant term of their result for the correlation energy. In the high-temperature case this exchange contribution is of order $\Lambda\gamma$, and hence smaller than the sum of the direct-interaction ring terms by a factor of γ. The ring diagrams with three or more particles and one exchange interaction are divergent, but the divergence is of lower order than the corresponding direct-interaction terms. Since the exchange correlation energy and pressure are small in comparison with the Debye–Hückel term in the near-classical limit (i.e. for high temperature and low density so that $\gamma \ll 1$), the detailed treatment of exchange interactions in the ring terms will be reserved for a later paper in connexion with the more complete evaluation of the direct ring interactions, equation (69).

5. APPLICATION TO A MULTICOMPONENT GAS

We consider now a real, fully-ionized gas composed of $\bar{N}_e$ electrons and $\bar{N}_i$ ions of charge z_i, mass M_i and spin s_i. Electrical neutrality requires $\bar{N}_e = z_i\bar{N}_i$. We may assume either Fermi or Bose statistics for the ions by using $f^-(p_i)f^+(p_i')$ or $b^-(p)b^+(p_i)$ in the expressions for the Q's. Since only the ring diagrams will be used, no account is taken of the sequence of bound states possible for the electron-ion system. The grand partition function is:

$$\exp\left(\frac{\Omega(\alpha_e,\alpha_i)}{kT}\right) = \sum_{N_eN_i} Tr\exp(\alpha_eN_e + \alpha_iN_i - \beta H_{N_eN_i}) \cong Z_0(\alpha_0)Z_0(\alpha_i)\exp\left(Q_1 + \sum_{n=2} Q_n\right), \quad (106)$$

where $\alpha_e = \beta\mu_e$ and $\alpha_i = \beta\mu_i$ are the chemical potentials of the electrons and ions, respectively. The Q_n's now refer to all possible clusters of n_e electrons and n_i ions so that $n_e + n_i = n$. The chemical potentials, α_{e0} and α_{i0}, of the non-interacting particles are defined by:

$$\zeta_e \equiv \frac{(2\pi\hbar)^3\rho_e}{(2s_e+1)\pi^{3/2}\bar{p}_e^{\,3}} = \mathscr{J}_{1/2}(\alpha_{e0})$$

$$\zeta_i \equiv \frac{(2\pi\hbar)^3\rho_i}{(2s_i+1)\pi^{3/2}\bar{p}_i^{\,3}} = \begin{cases}\mathscr{J}_{1/2}(\alpha_{i0}) & \text{F.D.}\\ \mathscr{J}_{1/2}(\alpha_{i0}) & \text{B.E.,}\end{cases} \quad (107)$$

where

$$\rho_e = \bar{N}_e/V, \qquad \rho_i = \bar{N}_i/V$$
$$\bar{p}_e = (2mkT)^{1/2}, \qquad \bar{p}_i = (2M_ikT)^{1/2}.$$

The pressure of the two component gas in the ring-diagram approximation is given by:

$$\frac{\Omega(\alpha_e,\alpha_i)}{kT} = \bar{N}_e\zeta_e^{-1}\mathscr{J}_{3/2}(\alpha_e) + \bar{N}_i\zeta_i^{-1}\begin{Bmatrix}\mathscr{J}_{3/2}(\alpha_i)\\ \mathscr{J}_{3/2}(\alpha_i)\end{Bmatrix} + Q_1 + \sum_{n=2} Q_n, \quad (108)$$

and the two chemical potentials are determined by normalizing to $\bar{N}_e$ and $\bar{N}_i$ according to:

$$\bar{N}_e = \frac{\partial}{\partial\alpha_e}\left(\frac{\Omega(\alpha_e,\alpha_i)}{kT}\right)_\beta,$$

$$\bar{N}_i = \frac{\partial}{\partial\alpha_i}\left(\frac{\Omega(\alpha_e,\alpha_i)}{kT}\right)_\beta. \quad (109)$$

Equations (109) reduce to (107) for no interaction, $e^2 = 0$.

The treatment of the ring diagrams will be illustrated for the case $n = 2$. In the place of one diagram for two electrons, Fig. 2(a), we now have the three diagrams of Fig. 5 which show the electron-electron, electron-ion and ion-ion interactions. If $U(r)$ is taken equal to e^2/r, the interaction potential between electron and ion is given by $-z_iU(r)$, and for an ion-ion pair is by $z_i^2U(r)$. The contribution from the three diagrams of Fig. 5 is:

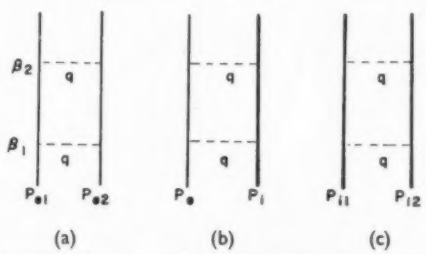


FIG. 5.—The diagrams for electron-electron, electron-ion and ion-ion interactions. The heavy lines indicate ions.

$$Q_2 = Q_{2ee} + Q_{2ei} + Q_{2ii},$$

where Q_{2ee} is given by (34) and Q_{2ei} is:

$$Q_{2ei} = z_i^2(2s_e+1)(2s_i+1) \times \left(\frac{V}{(2\pi\hbar)^3}\right)^3 \int d\mathbf{q}u(q)^2 \int d\mathbf{p}_e f^-(p_e)f^+(p_e') \times \int d\mathbf{p}_i \begin{Bmatrix}f^-(p_i)f^+(p_i')\\ b^-(p_i)b^+(p_i')\end{Bmatrix} R_2(\beta,D_{ei}), \quad (110)$$

where $R_2(\beta,D_{ei})$ is given by (33) with

$$D_{ei} = \Delta_e + \Delta_i = (p_e'^2 - p_e^2)/2m + (p_i'^2 - p_i^2)/2m.$$

The combinatorial factor of 1/2 in (34) does not occur

in Q_{2ei}, since the two particles are not identical. Each of the three diagrams of Fig. 5 may be analysed as in Section 2. For example, for Fig. 5(b) we have:

$$R_2(\beta, D_{ei}) = \int_0^\beta d\beta_2 \int_0^{\beta_2} dt_1\, e^{-D_{ei}t_1}$$
$$= \int_0^\beta d\beta_2 \int_0^{\beta_2} d\beta_1 e^{-(\beta_2-\beta_1)\Delta_e - (\beta_2-\beta_1)\Delta_i}, \quad (111)$$

from which we obtain:

$$Q_{2ei} = \frac{V}{(2\pi\hbar)^3}\int d\mathbf{q} z_i^2 u(q)^2 \frac{1}{2}\int_0^\beta \int_0^\beta d\beta_1\, d\beta_2$$
$$\times G_e(|\beta_2 - \beta_1|)\, G_i(|\beta_2 - \beta_1|), \quad (112)$$

where $G_i(\tau)$ is the ion propagator appropriate to either Fermi or Bose statistics:

$$G_1(q,\tau) = \frac{(2s_i+1)V}{(2\pi\hbar)^3}\int d\mathbf{p}_i \begin{Bmatrix} f^-(p_i)f^+(p_i') \\ b^-(p_i)b^+(p_i') \end{Bmatrix} e^{-\tau\Delta_i} \quad (113)$$

The eigenvalues of the integral equation (55), for $G_i(\tau)$ used as the kernel, are:

$$\lambda_{ik}(q) = \int_0^\beta d\tau\, e^{-2\pi i k T/\beta} G_i(q,\tau).$$
$$= V\rho_i\beta\mathscr{L}(q|\bar{p}_i, 2\pi k), \quad (114)$$

with $\mathscr{L}(q|\bar{p}_i, 2\pi k)$ given by (63). By using the expansion of the electron and ion propagators (55) in terms of the eigenvalues and eigenfunctions, we find that (112) becomes:

$$Q_{2ei} = \frac{V}{(2\pi\hbar)^3}\int d\mathbf{q} z_i^2 u(q)^2 \frac{1}{2}\sum_k \lambda_{ek}\lambda_{ik}. \quad (115)$$

Similarly, the ion-ion interaction gives:

$$Q_{2ii} = \frac{1}{2}\cdot\frac{V}{(2\pi\hbar)^3}\int d\mathbf{q} z_i^4 u(q)^2$$
$$\times \frac{1}{2}\int_0^\beta\int_0^\beta d\beta_1\, d\beta_2 G_i^2(|\beta_2-\beta_1|)$$
$$= \frac{1}{2}\frac{V}{(2\pi\hbar)^3}\int d\mathbf{q} z_i^4 u(q)^2 \frac{1}{2}\sum_k \lambda_{ik}^2\ . \quad (116)$$

Addition of the three contributions gives:

$$Q_2 = \frac{1}{2}\cdot\frac{V}{(2\pi\hbar)^3}\int d\mathbf{q} u(q)^2 \frac{1}{2}\sum_k (\lambda_{ek} + z_i^2\lambda_{ik})^2. \quad (117)$$

For an n-particle cluster composed of n_e electrons and n_i electrons arranged in a particular order on a ring, one must add the contributions from diagrams of all time orderings as was done in (54). The result analogous to (57) is $(-1)^n/2n\sum_k z_i^{2n_i}\lambda_{ek}^{n_e}\lambda_{ik}^{n_i}$, and this must be multiplied by $n!/n_e!n_i!$, the number of ways in which n_e electrons and n_i ions may be placed on a ring. Finally, when these terms of the binomial expansion are summed for all values of n_e and n_i so that $n_e + n_i = n$, the generalization of Q_n to a two-component gas is:

$$Q_n = \frac{V}{(2\pi\hbar)^3}\int d\mathbf{q} u(q)^n \frac{(-1)^n}{2n}\sum_k (\lambda_{ek} + z_i^2\lambda_i)^n. \quad (118)$$

The further generalization to a several-component gas obviously requires $(\Sigma_i z_i^2\lambda_{ik})^n$ in (118) where the summation is over every ionic species.

The ring-integral contribution to the pressure is:

$$(\Omega/kT)_{\text{ring}} = \frac{1}{2}\frac{V}{(2\pi\hbar)^3}$$
$$\times \int d\mathbf{q} \sum_{k=-\infty}^{\infty} [x_k(q) - \ln(1 + x_k(q))], \quad (119)$$

where

$$x_k(q) = u(q)\sum_i z_i^2\lambda_{ik}$$
$$= (4\pi e^2\beta\rho\hbar^2)q^{-2}\sum_i z_i^2 f_i \mathscr{L}(q|\bar{p}_i, 2\pi k)$$

with $\rho = \sum_i \rho_i$ and $f_i = \rho_i/\rho$.

Expression (119) is the ring-diagram contribution for a mixture of fermions and bosons for the entire temperature range. It will, of course, fail to be a valid approximation to the pressure as the temperature is reduced to the point where a significant number of electron and ion pairs are in hydrogenic-like bound states. We obtain near-classical-limit result from (119) as in Section 3, by approximating $\mathscr{L}(q|\bar{p}_i, 2\pi k)$ with its leading term, $\theta_i^2(\alpha_i)$, and neglecting all higher powers of q^2. The generalization of (74) to the multicomponent system, including the ideal gas terms and the direct-exchange term for electrons, is:

$$\frac{\Omega}{kT} = \sum_i \bar{N}_i\zeta_i^{-1}\mathscr{I}_{3/2}(\alpha_i)$$
$$+ \frac{\bar{N}_e}{2(2s_e+1)}\left(\frac{4\pi e^2\rho_e\beta\hbar^2}{\bar{p}_e^{\,2}}\right)\zeta_e^{-2}$$
$$+ \int_{-\infty}^{\alpha_e} d\alpha' \mathscr{I}^2_{-1/2}(\alpha') + \frac{2}{3}\bar{N}\Lambda\phi^{3/2} \quad (120)$$

where

$$\bar{N} = \sum_i \bar{N}_i$$
$$\Lambda = \sqrt{\pi}\, e^3\beta^{3/2}\rho^{1/2}$$
$$\phi = \sum_i z_i^2 f_i \theta_i^2(\alpha_i)\ . \quad (121)$$

The ring contribution in (119) in this limit is:

$$\left(\frac{\Omega}{kT}\right)_{\text{ring}} = \frac{V}{12\pi\lambda^3} = \frac{\bar{N}}{12\pi\rho\lambda^3}, \tag{122}$$

and, similarly, the potential energy due to the correlations of the ring interactions is, with use of (7), given by:

$$\frac{\bar{U}}{kT} = -\frac{V}{8\pi\lambda^3}, \tag{123}$$

where λ is the screening length generalized to include the effects of quantum statistics:

$$\lambda^{-2} = 4\pi e^2\beta \sum_i z_i^2\theta_i^2\rho_i. \tag{124}$$

This form of the screening length also comes into the Debye–Hückel theory when modified with Fermi and Bose statistics in place of Boltzmann statistics (KIDDER and DEWITT, to be published). The effective ionic charge in (124) is $z_i e\theta_i$, and, hence, less than the true charge $z_i e$ for Fermi statistics and greater for Bose statistics.

In order to obtain formulae for the pressure and internal energy of the multicomponent gas in terms of density (i.e. through α_{i0}) instead of the activities, it is necessary to find the dependence on the coupling constant for each α_i. As before, in Section 3, we assume for each species that α_i is the sum of α_{i0} and $\delta\alpha_i$, and find $\delta\alpha_i$ by using (109), which gives:

$$\bar{N}_i = \bar{N}_i\zeta_i^{-1}\mathscr{I}_{1/2}(\alpha_i) + N\Lambda\phi^{1/2}\frac{\partial\phi}{\partial\alpha_i}, \tag{125}$$

where

$$\frac{\partial\phi}{\partial\alpha_i} = z_i^2 f_i \zeta_i^{-1}\mathscr{I}_{-3/2}(\alpha_i).$$

By expanding $\mathscr{I}_{1/2}(\alpha_{i0} + \delta\alpha_i)$ (or $\mathscr{I}_{1/2}(\alpha_{i0} + \delta\alpha_i)$, for Bose statistics) and solving for $\delta\alpha_i$ in (125), we find:

$$\delta\alpha_i = -\Lambda\phi^{1/2}z_i^2\left(\frac{\mathscr{I}_{-3/2}(\alpha_{i0})}{\mathscr{I}_{-1/2}(\alpha_{i0})}\right), \tag{126}$$

which reduces to (81) for a one-component gas. The expansion of the ideal-gas terms in (120) give

$$\sum_i \bar{N}_i\zeta_i^{-1}[\mathscr{I}_{3/2}(\alpha_{i0} + \delta\alpha_i) - \mathscr{I}_{3/2}(\alpha_{i0})] = \bar{N}\sum_i f_i\delta\alpha_i$$
$$= -N\Lambda\phi^{3/2} + \bar{N}\Lambda\phi^{1/2}\sum_i z_i^2 f_i\eta(\alpha_{i0}), \tag{127}$$

where $\eta(\alpha_{i0})$ is given by (85) and (86) for Fermi and Bose statistics, respectively. The final result for the pressure to order e^3 is given by:

$$PV/kT = \sum_i \bar{N}_i\zeta_i^{-1}\mathscr{I}_{3/2}(\alpha_{i0})$$
$$- \bar{N}_e\frac{1}{4}\gamma_e^2\theta_{e0}^2(1 - \eta_x(\alpha_{e0})) - \frac{1}{3}\bar{N}\Lambda\phi^{3/2}$$
$$+ \bar{N}\Lambda\phi^{1/2}\sum z_i^2 f_i\eta(\alpha_{i0}), \tag{128}$$

and, similarly, the internal energy is given by:

$$\bar{E}/kT = \frac{3}{2}\sum \bar{N}_i\zeta_i^{-1}\mathscr{I}_{3/2}(\alpha_{i0})$$
$$- \bar{N}_e\frac{1}{2}\gamma_e^2\theta_{e0}^2(1 - \eta_x(\alpha_{e0})) - \bar{N}\Lambda\phi^{3/2}$$
$$+ \frac{3}{2}\bar{N}\Lambda\phi^{1/2}\sum_i z_i^2 f_i\theta_{i0}^2\eta(\alpha_{i0}). \tag{129}$$

The ionic exchange terms have not been included, since they are smaller than the electron exchange term by a factor of m/M_i. These expressions (128) and (129) are valid only in the near-classical limit, i.e. for temperatures and densities such that no degeneracy parameter differs too much from 1. At lower temperatures, (119) must be evaluated more completely, and the contributions from bound states should be included. Also, exchange correlation terms which give a contribution of order $\Lambda\gamma$ become important. Because of the actual smallness of the electronic-to-ionic mass ratio, m/M_i, even if the electrons are slightly degenerate ($0{\cdot}9 < \theta_{e0}^2 < 1$), the ions will be almost completely non-degenerate. The ideal-gas term for each ion is $\zeta_i^{-1}\mathscr{I}_{3/2}(\alpha_{i0}) \cong 1$ and $\theta_{i0}^2 \cong 1$, so that the $\eta(\alpha_{i0})$ terms are negligible. Thus, the statistics modifications embodied in (128) and (129) are, in practice, important only for electrons, the ions being nearly classical. Within the limits of validity, however, these expressions are easy to use to obtain numerical results for the pressure and energy of a slightly degenerate gas. The dimensionless parameters ζ, γ_e^2, and Λ are given in terms of temperature, density and fundamental constants. The value of α_{e0} may be read from the MCDOUGALL and STONER tables, using $\zeta_e = \mathscr{I}_{1/2}(\alpha_{e0})$. Those authors have also tabulated values of $\mathscr{I}_{-1/2}(\alpha_{e0})$ and $\mathscr{I}_{-3/2}(\alpha_{e0})$ needed to calculate θ_{e0}^2 and $\eta(\alpha_{e0})$. Only the $\eta_x(\alpha_{e0})$ needed for the direct-exchange term is not available in numerical tables; the series expansion (99), however, gives it with sufficient accuracy in the near-classical limit.

6. DISCUSSION

In Section 2, a formal expression valid for the entire temperature range was derived for the contribution of the direct-interaction ring diagrams to the equation of state of a one-component quantum gas. From this formal expression, explicit formulae suitable for numerical computation of pressure and internal energy in the near-classical limit were obtained by an appropriate mathematical approximation, i.e. neglect of terms of $O(q)^2$. The limits of validity of these near-classical-limit expressions, (87)

VOL 2 196

for a one-component gas and (128) and (129) for a multicomponent gas, will now be considered. These formulae represent an incomplete evaluation of (69), resulting in (1) the neglect of quantum-mechanical diffraction effects, (2) the neglect of the exchange interactions in the ring diagrams and (3) the neglect of the innumerable non-ring interactions.

To estimate the validity of the near-classical-limit formulae, we first note that three lengths are sufficient to completely describe the one-component quantum gas. These lengths are the average interparticle separation $l = \rho^{-1/3}$, the classical Debye screening length $\lambda_D = (4\pi e^2\beta\rho)^{-1/2}$, and the thermal de Broglie wavelength $\lambda_{\text{th}} = \hbar(\beta/2m)^{1/2}$. From the ratios of these lengths, two independent dimensionless parameters are possible. For expressing the results of the theory, it was found convenient to use three quantities ζ, Λ, and γ, which, in terms of the fundamental lengths, are given by:

$$\begin{aligned}\Lambda &= \frac{1}{8\pi}\left(\frac{l}{\lambda_D}\right)^3,\\ \gamma &= \lambda_{\text{th}}/\lambda_D,\\ \zeta &= 4\pi^{3/2}(\lambda_{\text{th}}/\lambda_D)^3 \quad (\text{using } s = 1/2)\\ &= \frac{\pi^{1/2}}{2}\frac{\gamma^3}{\Lambda}.\end{aligned} \tag{130}$$

The ideal-gas term, $\zeta^{-1}\mathscr{I}_{3/2}(\alpha_0)$, is a function of ζ only, and this term goes to 1 in the classical limit, $\zeta \ll 1$. The sum of the ring-diagram contribution in the classical limit reduces to the Debye–Hückel term which is $O(\Lambda)$. If all interactions are taken into account, it is expected that in the classical limit defined by $\hbar \to 0$, the thermodynamic functions of a one-component gas may be completely expressed in terms of the one classical parameter Λ. Considerable work has already been done on the problems of classifying and summing non-ring diagrams for classical gases, particularly by McMillan and Mayer (1945), Haga (1953), Meeron (1958) and Friedman (1959). Their results indicate that the next contribution to the pressure beyond the Debye–Hückel term is of order $\Lambda^2 \ln \Lambda$ for $\Lambda \ll 1$. Consequently, in the near-classical region where some deviations due to quantum statistics may be seen (the θ_0^3 factor), the sufficient smallness of Λ may be used as a criterion for obtaining accurate results with only the ring diagrams. It should be noted that $\Lambda = 1$ is a critical value for the mathematics of the classical electron gas. The results of Berlin and Montroll (1952) indicate that the correlations became so pronounced that a phase change occurs with the gas being forced into a lattice.

At high temperature, the complete evaluation of the ring-diagram contribution in (69) may be expected to yield $\Lambda f(\gamma,\zeta)$. Because of the mathematical approximation of dropping $O(q^2)$ terms, we obtained only $f(\gamma,\zeta) = \theta_0^3$. In a later paper it will be shown that a complete evaluation of (69) yields an expansion of $f(\gamma,\zeta)$ in powers of γ. The higher powers of γ beyond the first, or near-classical, term are due to the wave nature of the particles, and may be called 'diffraction terms.' In addition, the calculation of contributions from exchange interactions in the ring diagrams will yield terms of order $\Lambda\gamma$, $\Lambda\gamma^2$, $\Lambda\gamma^3$, etc. Thus the Debye–Hückel term, $\Lambda\theta^3$, is sufficiently accurate only if γ is small. Noting that λ_{th} is proportional to $\beta^{1/2}$, l is equal to $\rho^{-1/3}$, and λ_D is proportional to $\rho^{-1/2}\beta^{-1/2}$, we see that the high-temperature and low-density limit for which the near-classical-limit formulae are rigorously accurate requires:

$$\lambda_{\text{th}} \ll l \ll \lambda_D.$$

The conditions for the validity of (87) may be taken as $\Lambda < a$ and $\gamma < b$, where a and b are appropriate upper limits. If we take $a \sim b \sim 0{\cdot}3$, then using (130) we have $\zeta_{\max} \cong b^3/a \cong 0{\cdot}08$, for which $\theta_{0,\min}^2 \cong 0{\cdot}9$. Thus, the near-classical-limit formulae apply only to a slightly degenerate gas, $0{\cdot}9 < \theta_0^2 < 1$.

For a multicomponent gas, say ionized hydrogen, it is not possible to express the thermodynamic functions in terms of only the one classical parameter Λ. Because of the series of bound states possible for electron-proton pairs, the calculation of the partition function must be performed quantum mechanically, and the contributions from bound states will appear in the non-ring terms. The considerations of the previous paragraph, however, apply to the ring terms in the near-classical-limit. The recent results of Friedman (1959) indicate that the leading non-ring contribution, which for a one-component gas is $O(\Lambda^2 \ln \Lambda)$, is $O[\Lambda^2 \ln(\Lambda/\gamma)]$ for a two-component gas and hence infinite as $\hbar \to 0$.

The near-classical-limit formulae, (128) and (129), are applicable to a gas such as hydrogen when the temperature is high enough to insure nearly complete ionization. This requirement determines the ratio Λ/γ, since:

$$\Lambda/\gamma = \frac{e^2 m^{1/2}}{\hbar(kT)^{1/2}} = \left(\frac{\text{Rydberg}}{kT}\right)^{1/2}.$$

If we take $(kT)_{\min} = \text{Ryd.}$, then the maximum density is determined from:

$$\gamma_{\max} = \frac{\hbar(2\pi e^2\rho/m)^{1/2}}{\text{Ryd.}} = (8\pi a_0^3\rho)^{1/2},$$

where a_0 is the Bohr radius. Hence, with $\gamma_{max} = b \sim 0{\cdot}3$ we find $\rho_{max} = b^2/8\pi a_0^3 \sim 0{\cdot}3 \times 10^{25}$ cm^{-3}. The exchange correlation and diffraction effects involving powers of γ are thus appreciable only for high densities, but will be significant for astrophysical applications.

Acknowledgement—The author wishes to thank DR. HOWARD LEVINE for many helpful discussions and for correcting a number of mistakes.

REFERENCES

BERLIN T. H. and MONTROLL E. W. (1952) *J. Chem. Phys.* **20,** 75.
BLOCH C. and DOMINICIS C. DE (1959) *Nucl. Phys.* **7,** 459.
BRITTIN W. E. and SAKAKURA A. Unpublished work at the University of Colorado.
CHANDRASEKHAR S. (1957) *An Introduction to Study of Steller Structure* p. 400, Dover Publications, New York.
FERRAR W. L. (1941) *Algebra* pp. 22, 24. Oxford University Press.
FRIEDMAN H. L. (1959) *Molecular Phys.* **2,** 23, 190.
GELL-MANN M. and BRUECKNER K. A. (1957) *Phys. Rev.* **106,** 364.
GLASSGOLD A. E., HECKROTTE W. and WATSON K. M. *Phys. Rev.* **115** 1374 (1959).
HAGA E. (1953) *J. Phys. Soc. Japan* **8,** 714.
KIDDER R. E. and WITT H. E. DE To be published.
LANDAU L. O. and LIFSHITZ (1958) *Statistical Physics* pp. 105–7, Addison-Wesley, New York.
LEE T. O. and YANG C. N. (1959) *Phys. Rev.* **113,** 1165.
MCMILLAN W. G. and MAYER J. E. (1945) *J. Chem. Phys.* **10,** 629.
MÄCKE W. (1950) *Z. Naturf.* **5a,** 192.
MATSUBARA T. (1955) *Progr. Theor. Phys., Osaka* **14,** 351.
MEERON E. (1958) *Phys. Fluids* **1,** 139.
MONTROLL E. W. and MAYER J. E. (1941) *J. Chem. Phys.* **18,** 626.
MONTROLL E. W. and WARD J. C. (1958) *Phys. Fluids* **55.**
SAWADA K., BRUECKNER K. A., FUKADA N. and BROUT R. (1958) *Phys. Rev.* **108,** 507.
TRICOMI F. G. (1957) *Integral Equations* pp. 124–7, Interscience, New York.
WHITTAKER E. T. and WATSON G. N. (1952) *Modern Analysis* (4th Ed.) p. 77, Cambridge University Press.

APPENDIX A

Expansion of $R_n(\beta)$

The complex integral $R_n(\beta)$ occurring in the Q_n expression, (36), may be reduced to a sum of the $R_2(\beta)$ functions by expanding the product in the integrand in partial fractions:

$$\prod_i^{n-1} \frac{1}{(w - D_i)} = \sum_i^{n-1} \frac{a_i}{(w - D_i)}$$

$$a_i = \prod_{j \neq i} \frac{1}{(D_i - D_j)}. \tag{A.1}$$

Consequently, (37) becomes:

$$\begin{aligned} R_n(\beta) &= \sum_i^{n-1} a_i R_2(\beta, D_i) \\ &= \sum_i^{n-1} \frac{a_i}{D_i^2}(e^{-\beta D_i} - 1 + \beta D_i) \\ &= \sum_i^{n-1} a_i \left(\frac{\beta^2}{2!} - \frac{\beta^3 D_i}{3!} + \frac{\beta^4 D_i^2}{4!} - \ldots\right) \\ &= \sum_{r=2} \frac{(-1)^r \beta^r b_{nr}}{r!}, \end{aligned} \tag{A.2}$$

where

$$b_{nr} = \sum_i^{n-1} a_i D_i^{r-2}. \tag{A.3}$$

The expansion coefficient may easily be found for all r with the help of the Vandermonde determinant which, with $n-1$ factors, $D_1 \ldots D_{n-1}$, is:

$$\mathscr{D}_{n-1} = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ D_1 & D_2 & \ldots & D_{n-1} \\ D_1^2 & D_2^2 & & D_{n-1}^2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ D_1^{n-2} & D_2^{n-2} & \ldots & D_{n-1}^{n-2} \end{vmatrix} = \prod_{i<j}^{n-1} (D_i - D_j) \tag{A.4}$$

The minors of $\mathscr{D}_{n-1}$, when expanded about the bottom row, are also Vandermonde determinants:

$$\mathscr{D}_{n-1} = \sum_i^{n-1} (-1)^{n-1+i} \mathscr{D}_{n-2}(D_i) D_i^{n-2}, \tag{A.5}$$

where $\mathscr{D}_{n-2}(D_i)$ is the Vandermonde determinant of $n-2$ factors with D_i excluded. The partial-fraction coefficient in (A.1) is:

$$a_i = \frac{\mathscr{D}_{n-2}(D_i)}{\mathscr{D}_{n-1}},$$

so that (A.3) becomes:

$$b_{nr} = \mathscr{D}_{n-1}^{-1} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ D_1 & D_2 & & D_{n-1} \\ D_1^2 & D_2^2 & & D_{n-1}^2 \\ \vdots & \vdots & & \vdots \\ D_1^{n-3} & D_2^{n-3} & & D_{n-1}^{n-3} \\ D_1^{r-2} & D_2^{r-2} & \ldots & D_{n-1}^{r-2} \end{vmatrix}. \tag{A.6}$$

The numerator of (A.6) vanishes when the bottom row is identical with any other row; hence

$$\begin{aligned} b_{nr} &= 0 \qquad r < n \\ b_{nn} &= \mathscr{D}_{n-1}/\mathscr{D}_{n-1} = 1, \end{aligned} \tag{A.7}$$

and the expansion of $R_n(\beta)$ must begin with $O(\beta^n)$. If we define

$$Q_{n-1}(x) = \prod_i^{n-1} (x - D_i),$$

then it can be shown (FERRAR, 1941) that:

$$\frac{1}{Q_{n-1}(x)} = \mathscr{D}_{n-1}^{-1} \begin{vmatrix} 1 & \dots . & 1 \\ D_1 & & D_{n-1} \\ \vdots & & \vdots \\ D_1^{n-3} & & D_n^{n-3} \\ (x-D_1)^{-1} & \dots . & (x-D_{n-1})^{-1} \end{vmatrix}. \quad \text{(A.8)}$$

By putting $x = 1/y$ and expanding both sides of (A.8) in powers of y, one finds that the coefficients y^{n-2+p} on the right-hand side are the b_{nr} of (A.6) with $r = n - 2 + p$. The coefficients of y^{n-2+p} on the left-hand side are ${}_{n-1}H_p$, the sum of homogeneous products of $D_1, D_2, \dots, D_{n-1}$ of order p. Hence:

$$b_{np\,n-2+p} = {}_{n-1}H_p,$$

or

$$b_{n,n+1} = \sum_i^{n-1} D_i$$

$$b_{n,n+2} = \sum_i^{n-1} D_i^2 + \sum_{i<j} D_i D_j. \quad \text{(A.9)}$$

In the zero-temperature limit, $\beta \to 0$, $R_n(\beta)$ is linear in β, and (A.2) becomes:

$$R_n(\beta) = \beta \sum_i^{n-1} \frac{a_i}{D_i} + \sum_i^{n-1} \frac{a_i}{D_i^2}(e^{-\beta D_i} - 1)$$

$$\cong \beta b_{n-1,-1} \quad \text{for } \beta \text{ large.} \quad \text{(A.10)}$$

The coefficient $b_{n-1,-1}$ is given by the right-hand side of (A.8) with $x = 0$, and hence from the left-hand side we find:

$$b_{n-1,-1} = (-1)^n \prod_i^{n-1} \frac{1}{D_i}. \quad \text{(A.11)}$$

APPENDIX B

Evaluation of the direct exchange integral

The contribution to $\beta\Omega$ from the simple exchange interaction depends on the integral defined by (91). Since this integral is typical of the types of integrals encountered, some of the steps in its evaluation will be given here. In (91) one set of angular integration, say over $\mathbf{y}_1$, may be done immediately to give 4π, and the integration of the azimuthal angle of $\mathbf{y}_2$ gives 2π, so that (91) becomes:

$$\mathscr{I}_x(\alpha) = 8\pi^2 \int_0^\infty \frac{dy_1 y_1^2 f^-(y_1)}{\pi^{3/2}} \int_0^\infty \frac{dy_2 y_2^2 f^-(y_2)}{\pi^{3/2}} \times \int_0^\pi \frac{d(\cos\theta)}{y_1^2 + y_2^2 - 2y_1 y_2 \cos\theta}$$

$$= \frac{8}{\pi} \int_0^\infty dy_1 y_1 f^-(y_1) \times \int_0^\infty dy_2 y_2 f^-(y_2) \log\left|\frac{y_1 + y_2}{y_1 - y_2}\right|. \quad \text{(B.1)}$$

By expanding $f^-(y_1)f^-(y_2)$ in powers of e^α we have:

$$\mathscr{I}_x(\alpha) = \sum_{r_1=1, r_2=1} (-1)^{r_1+r_2} e^{(r_1+r_2)\alpha} \mathscr{I}_{x r_1 r_2} \quad \text{(B.2)}$$

$$\mathscr{I}_{x r_1 r_2} = \frac{8}{\pi} \int_0^\infty dy_1 y_1 e^{-r_1 y_1^2} \times \int_0^\infty dy_2 y_2 e^{-r_2 y_2^2} \log\left|\frac{y_1 + y_2}{y_1 - y_2}\right|. \quad \text{(B.3)}$$

After the variables are changed to $v^2 = r_2 y_2^2$, the y_2 integration gives:

$$\int_0^\infty dy_2 y_2 e^{-r_2 y_2^2} \log\left|\frac{y_1 + y_2}{y_1 - y_2}\right| = \frac{1}{r_2} \int_0^\infty dv v e^{-v^2} \log\left|\frac{\sqrt{r_2} y_1 + v}{\sqrt{r_2} y_1 - v}\right| = \frac{\sqrt{\pi}}{2} e^{-r_2 y_1^2} \operatorname{erfi}(\sqrt{r_2} y_1),$$

where $\operatorname{erfi}(a)$ is the imaginary error function, $\int_0^a dt\, e^{t^2}$. Consequently, $\mathscr{I}_{x r_1 r_2}$ becomes:

$$\mathscr{I}_{x r_1 r_2} = \frac{8}{\sqrt{\pi} r_2} \int_0^\infty dy_1 y_1 e^{-(r_1 + r_2) y_1^2} \int_0^{\sqrt{r_2} y_1} dt\, e^{t^2} = \frac{4}{\sqrt{\pi} r_2} \int_0^\infty dy_1 y_1 e^{-(r_1 + r_2) y_1^2} \int_0^{y_1} du\, e^{r_2 u^2}. \quad \text{(B.4)}$$

The remaining integrations may be performed most conveniently by changing the order of the integration with the help of the Dirichlet inversion formula (WHITTAKER and WATSON, 1952):

$$\mathscr{I}_{x r_1 r_2} = \frac{8}{(\pi r_2)^{1/2}} \int_0^\infty du\, e^{r_2 u^2} \int_u^\infty dy_1 y_1 e^{-(r_1 + r_2) y_1^2} = \frac{8}{(\pi r_2)^{1/2}} \int_0^\infty du\, e^{r_2 u^2} \frac{e^{-(r_1 + r_2) u^2}}{2(r_1 + r_2)} = \frac{2}{(r_1 + r_2)(r_1 r_2)^{1/2}}. \quad \text{(B.5)}$$

Hence:

$$\mathscr{I}_x(\alpha) = 2 \sum_{r_1=1, r_2=1} \frac{(-1)^{r_1+r_2} e^{-(r_1+r_2)\alpha}}{(r_1 + r_2)(r_1 r_2)^{1/2}}. \quad \text{(B.6)}$$

We may express the double summation as an integral by noting that the series expansion of $\mathscr{I}^2_{-1/2}(\alpha')$ is:

$$\mathscr{I}^2_{-1/2}(\alpha') = \sum_{r_1=1, r_2=1} \frac{(-1)^{r_1+r_2} e^{-(r_1+r_2)\alpha'}}{(r_1 r_2)^{1/2}}.$$

Hence, (B.6) is:

$$\mathscr{I}_x(\alpha) = \int_{-\infty}^{\alpha} d\alpha' \mathscr{I}^2_{-1/2}(\alpha'). \quad \text{(B.7)}$$

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 46 to 50. Pergamon Press Ltd. Printed in Northern Ireland

PHONONS AND PLASMONS*

R. BROUT
Department of Physics, Cornell University, Ithaca, New York

Abstract—The interaction of a heavy mass positive plasma (mass M) with a light mass negative plasma (mass m) is examined in the random phase approximation. The problem is solved in this approximation by a normal mode analysis, leading to a dispersion relation which is analysed in detail. In the limit of low momentum transfer q, the results are identical to those of BARDEEN (1937), BOHM and STAVER (1952) and BARDEEN and PINES (1955) corresponding to Coulomb interaction screened out with the Fermi–Thomas screening factor. Phonons arise with $c = \sqrt{m/3M}\, v_f$; the plasma frequency is $\sqrt{(4\pi ne^2/\mu)}$ where μ is the effective mass; v_f = the Fermi velocity; n = the electron density; c = the sound velocity. The phonon decays into electron-hole pairs with a relaxation time given by conventional perturbation theory using a screened electron-ion interaction. The screening arises in terms of a renormalized coupling constant, using a conventional field theoretic analysis.

At high-momentum transfer ($q > 2p_f$), the phonon is stable (i.e. the imaginary part of the dispersion relation as a function of complex variable vanishes) and the analytic character of the spectrum changes. This effect, as pointed out by KOHN (1959) in another context, can lead to determination of the Fermi surface in metals. Its pertinence to the present theory was pointed out by B. KNIGHT.

1. INTRODUCTION

IT is the purpose of this note to explore the consequences of electron-ion coupling in metals via Coulomb interaction in the same approximation (hereafter called random phase approximation or RPA) that has been used for the pure electron case (GELL-MANN and BRUECKNER, 1957; SAWADA, 1957; SAWADA, BRUECKNER, FUKUDA and BROUT, 1957; BROUT, 1957). The electron gas is considered free (i.e. Bloch states are plane waves) and the coupling to the ions is taken to be derivative in character, corresponding to coupling to the longitudinal phonons. In this paper we shall concern ourselves with the electron gas in the ground state only; phonons however may be present. The technique used is the normal mode analysis (of SAWADA, 1957; SAWADA, BRUECKNER, FUKUDA and BROUT, 1957; BROUT, 1957) and the main point of interest is the modification of the spectrum due to interaction with the ions. The results obtained do not differ in physical content from the work of BARDEEN (1937), BOHM and STAVER (1952) and BARDEEN and PINES (1955). It is hoped, however, that the mathematical methods here used lend more clarity to the problem than was obtained previously. Further, the phonon spectrum possesses some interesting features in the present work which is not immediately evident in previous work although it is easily obtainable by those methods). For small wave number, the phonon energy lies in the continuum of one electron excitation. For this reason its 'energy' is complex and a single phonon represents an approximate eigenstate in the sense of GLASER and KALLEN (1957) in that it decays with an inverse lifetime which is much less than its frequency. For high wave number phonons ($q > 2p_F$) where q is the phonon wave number and p_F the Fermi momentum) the analytic character of the spectrum changes and the phonon is stable against single electron decay. This effect was noted in another context independently by KOHN (1959) who pointed out the utility of this property in determining the Fermi surface by measuring the phonon spectrum (by neutron spectroscopy for example).

2. MATHEMATICAL DEVELOPMENT

The Hamiltonian adopted consists of three parts, the ion Hamiltonian H_{ion}, the electron Hamiltonian H_{el}, and the ion electron coupling, H_{int} (for simplicity we take one electron per ion). We consider them singly.

$$H_{\text{ion}} = \tfrac{1}{2}\sum_{\mathbf{q}}(M/q^2)[|\dot{\rho}_{\mathbf{q}}|^2 + \omega_M{}^2|\rho_{\mathbf{q}}|^2], \qquad (1)$$

where

$$\rho_{\mathbf{q}} = \rho_{-\mathbf{q}}{}^* = (1/\sqrt{N})\sum_{i=1}^{N}\exp\,[i\mathbf{q}\,.\,\mathbf{R}_i]$$
$$\cong (1/\sqrt{N})\sum \exp\,[i\mathbf{q}\,.\,\mathbf{R}_i{}^0][1 + i\mathbf{q}\,.\,\boldsymbol{\delta}_i],$$
$$= \rho_{\mathbf{q}}{}^0 + \delta_{\mathbf{q}} \qquad (2)$$

* This work was supported in part by the Office of Naval Research. The bulk of the work reported here was done in the summer of 1958 when the author was at the Bell Laboratories.

VOL
2
196

$\omega_M^2 = 4\pi ne^2/M$,

n = mean ion density = mean electron density,

ω_M = classical plasma frequency of the ions, of mass M,

$\mathbf{R}_i^0$ = equilibrium position of the ith ion,

$\boldsymbol{\delta}_i$ = deviation from equilibrium of the ith ion.

The form (1) is the standard Coulomb Hamiltonian of a system of charges. In (2) we introduce the linear approximation. In effect, this limits the ions to harmonic oscillations about their equilibrium positions with characteristic frequency ω_M. Only longitudinal oscillations are present in this approximation. A more detailed study of the problem reveals that this is a good approximation for small q, but for larger q, transverse modes are present with the sum rule due to KOHN (BROUT, 1959) $\sum_{i=1}^{3} \omega_i^2(\mathbf{q}) = \omega_M^2$. This may lead to interesting effects at higher q, but in the interest of simplicity will not be explored in this paper. Furthermore, the presence of ρ_q^0 in (2) will constitute a static potential for the electrons to move in. As we do not wish to be burdened by the complications of band structure, this term will be dropped. Thus, from now on, the symbol ρ_q will be used to designate the 'bare' longitudinal phonon δ_q. With this convention, we retain (1) as a harmonic oscillator potential with the commutation rules

$$[\dot{\rho}_q, \rho_{-q'}] = -i/M\, q^2 \delta_{qq'}. \tag{3}$$

We now turn to the electron Hamiltonian. Here we use the SAWADA formulation of the problem in second quantization

$$H_{\mathrm{el}} = T_{\mathrm{el}} + \sum_{\mathbf{q}} (2\pi e^2/q^2\Omega)[\sigma_{-\mathbf{q}}\sigma_{\mathbf{q}}] + \text{constant}, \tag{4}$$

T_{el} = kinetic energy of the electrons,

$$\sigma_q = \sum_{\substack{p \le p_F \\ |\mathbf{p}+\mathbf{q}|>p_F}} [d_{\mathbf{q}}(\mathbf{p}) + d_{-\mathbf{q}}{}^*(-\mathbf{p})] = \sigma_{-\mathbf{q}}{}^*,$$

$d_{\mathbf{q}}(\mathbf{p}) = a^*_{\mathbf{p}+\mathbf{q}} a_{\mathbf{p}}$,

$a_{\mathbf{p}} \equiv$ annihilation operator of an electron in state p.

The exact commutation rules of $d_q(p)$ are complicated and are not used in this paper, but only RPA

$$[d_{\mathbf{q}}(\mathbf{p}), d_{\mathbf{q}'}{}^*(\mathbf{p}')] = \delta_{\mathbf{q}\mathbf{q}'}\delta_{\mathbf{p}\mathbf{p}'}. \tag{5}$$

A full discussion of the steps leading to (4) and (5) as a model Hamiltonian for the high-density limit of the Coulomb problem is given by SAWADA (1957). The essential feature is that only virtual processes taking electrons to and from the Fermi sea are considered and these must be direct. Equation (5) excludes exchange.

For the interaction Hamiltonian we adopt

$$H_{\mathrm{Int}} = -\sum_{\mathbf{q}} (2\pi e^2/q^2\Omega)[\sigma_{-q}\rho_q + \sigma_q\rho_{-q}]. \tag{6}$$

This is considered in RPA as well, so that the commutation rules (3) and (5) are adopted in connexion with (6). This is a drastic step. The previous approximations are all founded on strong theoretical argument, whereas the last is unfounded. In particular, it leaves superconductivity out of the picture. The virtue of the present method is that it provides a simple model whose consequences are easily derivable. The results are qualitatively those of a normal metal and yield amazingly good quantitative agreement for the alkali metals (BOHM and STAVER, 1952; BARDEEN and PINES, 1955). It is very much an open question to explain why in some cases the present method is qualitatively correct (alkalis and other normal metals) and in other cases incorrect (superconductors). PINES (1958) has taken some steps in this direction, but much remains to be done.

In summary, our total model Hamiltonian is the sum of (1), (4) and (6) conveniently written as

$$H = T_{\mathrm{ion}} + T_{\mathrm{el}} + \sum_{\mathbf{q}} (2\pi e^2/q^2\Omega)|\sigma_q - \rho_q|^2 + \text{constant}, \tag{7}$$

together with the commutation rules (3) and (6). In the form (7), it is seen that the approximate Hamiltonian strongly resembles the initial Coulomb Hamiltonian $V_{\mathrm{Coulomb}} = \sum_{\mathbf{q}}(2\pi e^2/q^2\Omega)|(\rho_{\mathbf{q}})_{\mathrm{ion}} - (\rho_{\mathbf{q}})_{\mathrm{el}}|^2$ where $\sigma_{\mathbf{q}}$ replaces $(\rho_{\mathbf{q}})_{\mathrm{el}}$ in the approximate Hamiltonian. $(\rho_{\mathbf{q}})_{\mathrm{el}}$ is given by

$$(\rho_{\mathbf{q}})_{\mathrm{el}} = \sum_{\mathbf{p}} [d_{\mathbf{q}}(\mathbf{p}) + d_{-\mathbf{q}}{}^*(-\mathbf{p})], \tag{8}$$

and hence includes all electron virtual processes. The other 'multilation' is the adoption of the commutation rule (5).

Since the problem at hand is now linear, its solution is trivial. It is convenient to define the variable $\tau_{\mathbf{q}}$

$$\tau_{\mathbf{q}} = (\rho_q + i\omega_M{}^{-1}\dot{\rho}_q); \quad \rho_q = \tfrac{1}{2}(\tau_q + \tau_{-q}{}^*), \tag{9}$$

We then have (with $\hbar = 1$) the equations of motion

$$i\dot{\tau}_{\mathbf{q}} = \omega_M \tau_{\mathbf{q}} - \omega_M \sigma_{\mathbf{q}},$$

$$i\dot{d}_q(p) = \omega_{\mathbf{q}}(\mathbf{p})\, d_{\mathbf{q}}(\mathbf{p}) + (4\pi e^2/q^2\Omega)(\sigma_{\mathbf{q}} - \rho_{\mathbf{q}}), \tag{10}$$

where $\omega_{\mathbf{q}}(\mathbf{p}) = \varepsilon(\mathbf{p}+\mathbf{q}) - \varepsilon(\mathbf{p})$; $\varepsilon(p) = p^2/2m$.

These coupled equations are diagonalized by the normal modes

$$\xi_{\mathbf{q}}(\omega) = [\eta_{\mathbf{q}}(\omega) - \tau_{\mathbf{q}}(\omega)/\omega][N_{\mathbf{q}}(\omega)]^{-\frac{1}{2}}, \qquad (11)$$

where

$$\eta_{\mathbf{q}}(\omega) = \sum_{\mathbf{p}} [d_{\mathbf{q}}(\mathbf{p})/(\omega - \omega_{\mathbf{q}}(\mathbf{p}) + d_{-\mathbf{q}}{}^{*}(-\mathbf{p})/(\omega + \omega_{\mathbf{q}}(\mathbf{p}))],$$

$$\tau_{\mathbf{q}}(\omega) = \rho_{\mathbf{q}} + i\dot{\rho}_{\mathbf{q}}/\omega,$$

if the new frequencies ω, satisfy the dispersion relation

$$1 - \omega_M{}^2/\omega^2 - \varphi_{\mathbf{q}}(\omega) \equiv \Phi_{\mathbf{q}}(\omega) = 0, \qquad (12)$$

$$\varphi_{\mathbf{q}}(\omega) = [4\pi e^2/q^2\Omega] \sum_{\substack{|\mathbf{p}|>p_F \\ |\mathbf{p}+\mathbf{q}|>p_F}} [1/(\omega - \omega_{\mathbf{q}}(\mathbf{p})) - 1/(\omega + \omega_{\mathbf{q}}(\mathbf{p}))]. \qquad (13)$$

There is a separate dispersion relation for each $\mathbf{q}$ corresponding to the diagonalization of the $\mathbf{q}^{\text{th}}$ Hamiltonian.

The normalization factor N which appears in (11) is easily computed by direct evaluation to be

$$N_{\mathbf{q}}(\omega) = [d\Phi_{\mathbf{q}}(\omega)/d\omega]. \qquad (14)$$

The transformation (11) is unitary, so that (14) is all that is needed to compute the energy. In fact calling $V_{\mathbf{q}} = (2\pi e^2/q^2\Omega)|\rho_{\mathbf{q}} - \sigma_{\mathbf{q}}|^2$, it is easy to show that

$$\langle V_q \rangle = \tfrac{1}{2} \sum_{\omega} [N_q(\omega)]^{-1}. \qquad (15)$$

Integration over the coupling constant e^2 gives the total energy (SAWADA, 1957). As this is of no great interest, we shall turn to a discussion of the dispersion relation (12) its most interesting aspect being collective co-ordinates.

As it stands, the roots of equation 12 form a quasi-continuum of roots which are infinitesimally close to the unperturbed $\omega_{\mathbf{q}}(\mathbf{p})$. For small $q(q < q_c)$ there is one root far removed from the boundary of the continuum. This is the plasma frequency. To derive the plasma frequency, note that for large ω, $\varphi(\omega) \to \omega_m{}^2/\omega^2$, so that

$$\Phi_{\mathbf{q}}(\omega) \xrightarrow[q \ll q_c]{\omega \gg \omega_{\mathbf{q}}(\mathbf{p}_F)} 1 - \omega_M{}^2/\omega^2 - \omega_m{}^2/\omega^2. \qquad (16)$$

Here we have introduced the classical plasma frequency of the electron gas of mass m

$$\omega_m{}^2 = 4\pi n e^2/m, \qquad (17)$$

so that

$$\omega_{Pl}{}^2 = \omega_m{}^2 + \omega_M{}^2 = 4\pi n e^2/\mu. \qquad (18)$$

where $1/\mu = 1/m + 1/M =$ (reduced mass)$^{-1}$. It is thus seen as expected that the plasma mode is an optical mode with the two plasmas moving in opposite directions, giving rise to the reduced mass. This is, of course, directly derivable on macroscopic grounds. The schematic dispersion relation for small q is sketched in Fig. 1. The infinite asymptotes are at $\omega_{\mathbf{q}}(\mathbf{p})$. The set of curves represents the function $\varphi_{\mathbf{q}}(\omega)$. Intersection with the horizontal line 1 gives the roots for the free electron case, and intersection with the curve gives the roots for the present case.

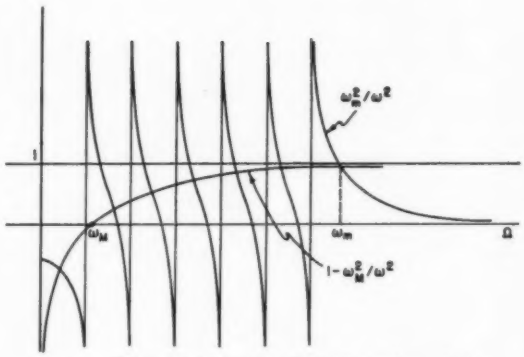


FIG. 1.—Schematic diagram of the dispersion relation $\Phi_q(\omega) = 0$.

Outside of the continuum the function $\varphi(\omega)$ may be evaluated by passing to the infinite limit, and converting sums to integrals. This then gives the dispersion relation for the plasmon.

Having arrived at an optical mode, one is tempted to ask for an accoustic mode. To do this we study $\varphi_{\mathbf{q}}(z)$ in the complex plane as a function of the complex variable performing all sums as integrals. One has immediately that $\varphi_{\mathbf{q}}(z)$ is analytic everywhere in the plane but on the real axis in the interval $-\omega_q(p_F) \leq x \leq \omega_q(p_F)$ where there is a branch cut. p_F is the Fermi momentum taken in the direction $\mathbf{q}$ so that $\omega_{\mathbf{q}}(p_F) = (p_F q + q^2/2)/m$ is the boundary of the continuum for fixed $\mathbf{q}$. We may now ask for the roots of $\Phi_{\mathbf{q}}(z) = 0$ in the complex plane. We postulate the existence of such a solution near the real axis and such that Re(z) lies within the continuum region. In this case Re$\varphi(z)$ is approximated by its principal value $\bar{\varphi}(x)$ and the dispersion relation is

$$1 - \omega_M{}^2/x^2 - \bar{\varphi}_{\mathbf{q}}(x) = 0. \qquad (19)$$

$\bar{\varphi}_{\mathbf{q}}(x)$ is plotted in Fig. 2 together with its intersection with $1 - \omega_M{}^2/x^2$. It is possible to integrate out and get $\bar{\varphi}_q(x)$ explicitly. This is not instructive, however, The salient features are: (1) $\lim_{x\to 0} \bar{\varphi}(x) = -\kappa^2/q^2$, where $\kappa^2 = (4\pi n e^2/2/3\varepsilon_F)$, κ^{-1} is the Fermi–Thomas screening length. (2) $\lim_{x\to 0} \bar{\varphi}'(x) = 0$. (3) There are four singular points at $\pm\omega_1, \pm\omega_2$ where $\omega_1 = (p_F q - q^2/2)/m$; $\omega_2 = (p_F q + q^2/2)/m$. $\bar{\varphi}(x)$ is finite at these points but possesses logarithmically infinite derivatives as shown in the diagram. (4) $\lim_{x\to\infty} \bar{\varphi}(x) = \omega_m{}^2/x^2$.

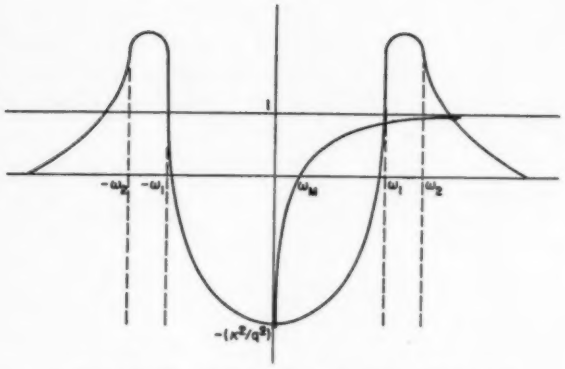


FIG. 2.—Schematic diagram of the dispersion relation $\bar{\Phi}_q(\omega) = 0, \quad q < 2P_F$.

As seen from Fig. 2 a solution of equation (19) is possible for small x. One then finds (for small q)

$$1 - \omega_M{}^2/x^2 + \kappa^2/q^2 = 0, \tag{20}$$

or for small q

$$x^2 = q^2/\kappa^2\, \omega_M{}^2 \equiv c^2q^2, \tag{21}$$

where we have introduced the speed of sound $c^2 = \omega_M{}^2/\kappa^2$. Using the definitions of ω_M, and κ one finds with $v_F = p_F/m$

$$c^2 = \tfrac{1}{3}(m/M]v_F{}^2, \tag{22}$$

a result due to STAVER. This result can also be derived from simple macroscopic argument. It may be written $c^2 = B/M$ where B is the bulk modulus of the free Fermi gas. In this form one sees that it is the compressibility of the Fermi gas, which follows the ions adiabatically which determines the phase velocity of the acoustic mode. In the form $x^2 = [q^2/\kappa^2)\omega_M{}^2$ one sees that ions whose motion is coherently excited with wave vector q, induces a response in the electron which screens out the inter-ion force from pure Coulomb to a screened Coulomb force, the screening constant being the familiar Fermi–Thomas screening factor. For a discussion of the remarkable agreement of (22) with observations on alkali metals, see BARDEEN and PINES (1955).

To justify the original ansatz that we have a good solution to the dispersion relation, it now must be shown that the phonon is indeed a good approximate eigenstate in the sense of GLASER and KALLÉN (1957). To show this we have followed their procedure and formed the wave packet corresponding to a phonon. The recipe is to let the phonon packet be formed by making the creation operator $\xi_q{}^*(\omega_{ph} + i\lambda)$ where ξ is given by equation (11) and ω_{ph} is the solution of (19). λ is a positive real number chosen such that $\lambda \gg \gamma$ where γ^{-1} is the lifetime of the phonon. One then calculates the expectation value of exp $[iHt]$ in $\xi_q{}^*|0\rangle$. The result of this calculation is (for $\gamma \ll \omega_{ph}$)

$$\langle 0|\xi_q{}^*(\omega_{ph} + i\lambda)|e^{iHt}|\xi_q(\omega_{ph} + i\lambda)|0\rangle = \exp\,[i\omega_{ph}t]\exp\,[-\gamma t/2] + 0(\gamma/\lambda). \tag{23}$$

If a frequency other than a solution of (19) is used the packet dissipates according to λ and in the limit of large λ nothing is left. On the other hand, if the principal value relation (19) is satisfied, one gets (23). This is the meaning of an approximate eigenstate. The quantity γ that appears in (23) is given by (we assume $\gamma/\omega_{ph} \ll 1$)

$$\gamma \cong [2\pi(4\pi c^2/q^2\Omega)\sum_{\mathbf{p}} \delta(\omega_{ph} - \omega_{\mathbf{q}}(\mathbf{p})]/d\bar{\Phi}/dx], \tag{24}$$

where $\bar{\Phi} = 1 - \omega_M{}^2/x^2 - \bar{\varphi}(x)$. The quantity $\bar{\Phi}'$ appearing in the denominator of (24) arises from the normalization factor $N(\omega)$ given by equation 14. Its effect here is to renormalize the coupling constant. For small q, we have $\bar{\varphi}'(\omega_{ph}) \cong 0$ and hence $\bar{\Phi}' = 2\,\omega_M{}^2/\omega_{ph}{}^3 = 2(\kappa^2/q^2)\omega_{ph}{}^{-1}$. Thus for small q

$$\gamma = \pi\omega_{ph}(4\pi e^2/\kappa^2\Omega)\sum_{\mathbf{p}} \delta(\omega_{ph} - \omega_{\mathbf{q}}(\mathbf{p})). \tag{25}$$

The effect of the renormalization factor $\bar{\Phi}'$ is to change the coupling from direct Coulomb to screened Coulomb coupling between the ions and the electrons as expected. Equation (24) is the result that would be derived in perturbation theory using a screened electron-ion interaction for H_{int} in place of (6) (i.e. Replace q^2 by κ^2 in (6) and pretend that ρ_q is a phonon). Our discussion of the phonon for small q terminates with the justification of the initial ansatz. Direct calculation gives $\gamma/\omega_{ph} = \sqrt{3/2}\sqrt{m/M} \ll 1$ which justifies the steps of our derivation.

It would appear from Fig. 2 that a second solution of $\bar{\Phi}(x) = 0$ exists in the continuum. It may be shown however that in this case Im $\Phi(z)$ is the same order of magnitude as Re$\Phi(z)$ so that $\bar{\Phi}(x)$ is a poor approximation to Re$\Phi(z)$. In fact no solution exists corresponding to the second root of $\bar{\Phi}$ in the continuum.

Finally we note that for large q, in particular for $q > 2p_F$, we have $-\omega_2 > \omega_1$ and the picture Fig. 2 changes to Fig. 3.† Now note that the phonon mode falls outside the continuum i.e. energy and momentum conservation do not allow the phonon to decay into single electron-hole pairs (of course, outside RPA the phonon can decay into more than one electron-hole pair but this will have much reduced probability). Finally the analytic character of the phonon spectrum changes abruptly at $q = 2p_F$ which is the effect pointed out by KOHN (1959).

† This effect was pointed out to the author by Mr. BRUCE KNIGHT to whom Fig. 3 is due.

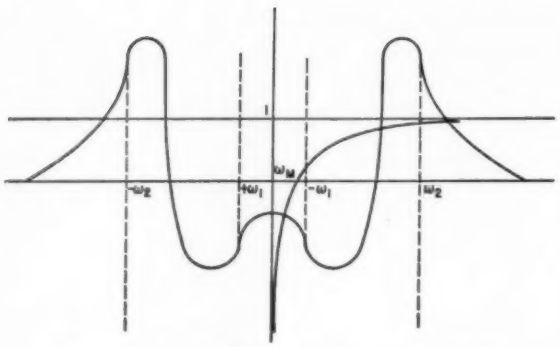


FIG. 3.—Schematic diagram of the dispersion relation $\overline{\Phi}_q(\omega) = 0, \quad q > 2P_F$.

3. CONCLUSION

This paper has presented a systematic discussion of electron-ion interactions in RPA. The physical results are (1) the existence of plasmon as an optical mode (2) the existence of the decaying phonon as an acoustic mode. The conventional perturbation picture of electron-ion interactions is recovered where the interactions is the screened Coulomb interaction of BARDEEN. For large q, the phonon becomes stable against one-particle decay and the analytic character of the spectrum changes at $q = 2p_F$.

Acknowledgment—I am indebted to MR. BRUCE KNIGHT for his very informative analysis of the analytic character of the dispersion relation.

REFERENCES

BARDEEN J. (1937) *Phys. Rev.* **52,** 688.
BARDEEN J. and PINES D. (1955) *Phys. Rev.* **99,** 1140.
BOHM D. and STAVER T. (1952) *Phys. Rev.* **84,** 836.
BROUT R. (1957) *Phys. Rev.* **108,** 515.
BROUT R. (1959) *Phys. Rev.* **113,** 43.
GELL-MANN M. and BRUECKNER K. (1957) *Phys. Rev.* **106,** 364.
GLASER W. and KALLEN G. (1957) *Nucl. Phys.* **2,** 706.
KOHN W. (1959) *Phys. Rev.* (*Letters*) **2,** 393.
PINES D. (1958) *Phys. Rev.* **109,** 280.
SAWADA K. (1957) *Phys. Rev.* **106,** 372; SAWADA K., BRUECKNER K., FUKUDA N. and BROUT R. (1957) *ibid.* **108,** 507.

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 51 to 56. Pergamon Press Ltd. Printed in Northern Ireland

COLLECTIVE OSCILLATIONS OF A SYSTEM OF INTERACTING FERMIONS

A. E. GLASSGOLD
Lawrence Radiation Laboratory, University of California, Berkeley, California

Abstract—Collective oscillations for an infinite system of fermions are discussed for the general case where the force between two particles depends on the internal variables as well as on their relative separation and momentum. Two theories are reviewed, one by WATSON, HECKROTTE and GLASSGOLD, which is based on SAWADA's treatment of the electron gas, and the other, a semiclassical treatment by LANDAU.

1. INTRODUCTION

THE quantum mechanical problem of the degenerate electron gas has received a great deal of attention in recent years. Significant progress has been made in calculating the ground state energy and properties of the excited states, particularly the collective aspects of this many-body system. There are, of course, other infinite systems of particles obeying Fermi-Dirac statistics which are of interest. As examples, we may mention the low-temperature behaviour of liquid He^3 or the popular abstraction called nuclear matter. In connexion with the collective motions, the novel aspect of such systems is that the internal variables, such as spin and isotopic spin, are now important. This occurs because the force between two helium atoms or between two nucleons depends strongly on the internal variables. On the other hand, the only force between two nonrelativistic electrons is the familiar spin-independent Coulomb repulsion. We are thus led to expect new kinds of collective oscillations for general fermion systems besides the familiar plasma oscillations of the electron gas.

Oscillations of this type were first considered by KLIMONTOVICH and SILIN (1952) and by FERENTZ, GELL-MANN and PINES (1953) in 1952. Starting in 1956, LANDAU (1956) and his school developed the theory of the Fermi liquid with He^3 in mind as the important application. As will be discussed below, this theory has marked classical and phenomenological aspects. A review of this work by ABRIKOSOV and KHALATNIKOV (1958) has recently appeared in translation. Independently, WATSON, HECKROTTE and the author (1959) developed a theory for the collective oscillations of fermion systems and applied it to the problem of nuclear matter. This treatment is a generalization of the theory of the electron gas as formulated by SAWADA (1957). As such, it is more of a microscopic treatment than is LANDAU's approach. In the rest of this report, the essential steps and results of this procedure will be reviewed. Then a brief discussion of LANDAU's theory will be made. A detailed discussion of the application of these theories to ^3He and nuclear matter will not be presented here. Such discussion may be found in the literature (ABRIKOSOV and KHALATNIKOV, 1958; GLASSGOLD, HECKROTTE and WATSON, 1959).

2. THE COMPLETE HAMILTONIAN

The introduction of more general kinds of forces leads to interesting new phenomena, but naturally at the expense of more complicated calculations. The difficulties arise from the simple increase in the number of degrees of freedom and from the complicated nature of the force law. To be more specific, the interaction between two particles in ^3He or nuclear matter, for example, involves a repulsive core for small separations with a relatively weak attractive region outside. In other words, the interaction is strongly momentum-dependent and also depends on spin (and isotopic spin).

In order to handle this difficult situation we adopt the following popular point of view. We assume that the states of the system bear some resemblance to the ideal Fermi gas in spite of the strong interactions.

To first approximation, we now have a system of non-interacting *quasiparticles* whose effective mass is different than the particle mass. The interaction between the quasiparticles is considerably weaker and better behaved than the actual force law between the particles. With reference to nuclear matter we are, of course, thinking of BRUECKNER's theory (BRUECKNER and GAMMEL, 1958).

We will let $P = (\mathbf{p},\lambda)$ label the state of a quasiparticle, where $\mathbf{p}$ is the momentum and λ is the set of internal variables such as spin, isotopic spin, etc. In the language of second quantization, a_P^+ and a_P

are the operators which create and destroy a quasiparticle in state P. These operators obey the anticommutation rules for fermions

$$[a_P, a_{P'}{}^+]_+ = \delta_{P,P'}, \qquad [a_P, a_{P'}]_+ = 0.$$

Here $P = P'$ requires every quantum number of the two states to be the same. In terms of these operators it is assumed that the Hamiltonian can be written as

$$H = E_B + \sum_P \varepsilon_P a_P{}^+ a_P - \sum_{P<P_0} \varepsilon_P + \frac{1}{2\Omega} \times \sum_{P_1P_2P_1'P_2'}{}' (p_1'P_2'|K|P_1P_2) a_{P_1}^+ a_{P_2}^+ a_{P_2'} a_{P_1'}. \quad (1)$$

The symbols which occur here have the following meanings: Ω is simply the volume of the quantization box; $P < P_0$ means that the sum is restricted to $p = |\mathbf{p}| < p_0$, where p_0 is the Fermi momentum:

$$\frac{4\pi}{3} p_0{}^3 \Omega = \frac{N}{S};$$

N is the total number of particles, S is the number of internal variables ($S = 2$ for ^{3}He, $S = 4$ for nuclear matter, etc.) and Planck's constant has been set equal to unity.

The interaction between the quasiparticles is given by the operator K, and its matrix elements depend on the states of the two particles before (P_1, P_2) and after (P_1', P_2') 'scattering.' The form of the matrix elements is restricted by various conservation laws, e.g. momentum conservation requires $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'$. (We shall not discuss here the way this operator is determined in a fundamental theory, and we may even regard it phenomenologically.) The constant E_B is the average value of H for the ground state ϕ_0 of the system of non-interacting quasiparticles, i.e. the state in which $a_P a_P{}^+ = 1$ for $P < P_0$ and 0 for $P > P_0$. Thus

$$E_B = \sum_{P<P_0} \varepsilon_P + \frac{1}{2\Omega} \sum_{P_1,P_2<P} [(P_1P_2|K|P_1P_2) - (P_1P_2|K|P_2P_1)]. \quad (2)$$

The prime on the second summation in equation (1) means that the terms diagonal for ϕ_0 have been excluded. Finally the energy of the quasiparticles is

$$\varepsilon_{P_1} = \varepsilon_{p_1} = p_1{}^2/2M + \frac{1}{\Omega} \sum_{P_2<P_0} [(P_1P_2|K|P_1P_2) - (P_1P_2|K|P_2P_1)].$$

It will be convenient to write ε_{p_1} near p_0 approximately as

$$\varepsilon_{p_1} \approx p_1{}^2/2M^*,$$

where M^* is the effective mass at the Fermi surface.

3. SAWADA'S APPROXIMATION

The procedure adopted for solving the above Hamiltonian is to break the interaction into two parts. One part can easily be diagonalized and has collective eigenstates. The other part presumably will cause only small effects when treated as a perturbation.

In order to define this separation we introduce *particle* and *hole operators:*

$$a_P = \begin{cases} \alpha_P & P > P_0 \\ \beta_P{}^+ & P < P_0 \end{cases} \qquad a_P{}^+ = \begin{cases} \alpha_P{}^+ & P > P_0 \\ \beta_P & P < P_0 \end{cases}$$

The momentum associated with a particle operator is always $>p_0$; the momentum associated with a hole is always $<p_0$. Next we combine these into *pair* creation and annihilation operators

$$\gamma_{P'P}^+ = \alpha_{P'}{}^+ \beta_P{}^+. \qquad \gamma_{P'P} = \beta_P \alpha_{P'}.$$

The first momentum p associated with a pair operator is always $>p_0$, the second p is always $<p_0$. (These conditions, associated with the definition of particle and holes, will not usually be written explicitly in what follows.)

If we now examine the potential energy of interactions of the quasiparticles,

$$\frac{1}{2\Omega} \sum_{P_1P_2P_1'P_2'} (P_1'P_2'|K|P_1P_2) a_{P_1}^+ a_{P_2}^+ a_{P_2} a_{P_1},$$

we can consider 16 classes of terms. These 16 classes arise from the two possibilities available for each of the 4 momenta, $p_1'p_2'p_2p_1$, i.e. each momentum is either $<p_0$ or $>p_0$. Of these 16 classes there are 6 which refer *only* to *pairs* as just defined, i.e. terms in which two momenta are $<p_0$ and two are $>p_0$. If V is the part of the interaction which is to be treated seriously and V' the part to be considered as a perturbation, then the 10 classes of terms which are *not* pair terms are certainly included in V'. Of the pair terms we include the four which can be written as

$$V = \frac{1}{2\Omega} \sum_{P_1P_2P_1'P_2'} (P_1'P_2'|K|P_1P_2) \times (\gamma_{P_1'P_1}^+ + \gamma_{P_1P_1'})(\gamma_{P_2'P_2}^+ + \gamma_{P_2P_2'}). \quad (3)$$

Each term includes operators which (1) create 2 pairs ($\gamma_{P_1'P_1}^+ \gamma_{P_2'P_2}^+$) (2) destroy 2 pairs ($\gamma_{P_1P_1'} \gamma_{P_2P_2'}$) and (3) transform one pair into another ($\gamma_{P_1P_1'} \gamma_{P_2'P_2}^+ + \gamma_{P_1'P_1} \gamma_{P_2P_2'}$). The exchange terms for this last type (in which P_2 and P_2' are interchanged) have not been included and appear instead in V'. (A certain amount of re-ordering has also been done in obtaining the symmetrical form of equation (3), and certain trivial additions have been made to V' to compensate.)

Finally, the kinetic energy is

$$K = \sum_P \varepsilon_P(\alpha_P^+\alpha_P - \beta_P^+\beta_P). \tag{4}$$

The complete Hamiltonian is

$$H = E_B + K + V + V', \tag{5}$$

where H, E_B, K and V are given by equations (1–4) and $V' = H - (E_B + K + V)$. We now solve the truncated Hamiltonian

$$H_0 = E_B + K + V \tag{6}$$

by making the further assumption that the pair operators $\gamma^+_{P'P}$ and $\gamma_{P'P}$ satisfy *boson commutation relations:*

$$[\gamma_{P_1'P_1},\gamma^+_{P_2'P_2}]_- = \delta_{P_1P_2}\delta_{P_1'P_2'}[\gamma_{P_1'P_1},\gamma_{P_2'P_2}]_- = 0. \tag{7}$$

With this assumption it can be shown that

$$[\gamma^+_{P_1'P_1},K]_- = -(\varepsilon_{P_1'} - \varepsilon_{P_1})\gamma^+_{P_1'P_1},$$

$$[\gamma^+_{P_1'P_1},V]_- = -\frac{1}{\Omega}\sum_{P_2P_2'}(P_1P_2'|K|P_1'P_2)(\gamma^+_{P_2'P_2} + \gamma_{P_2P_2'}).$$

From this result it follows that the excitation energies (and the operators creating the excited states from the ground state) can be determined exactly with standard matrix procedures.

If ψ_0 is the ground state of H_0, then it can be shown that the operator B_q^+, which creates the excited state $B_q^+\psi_0$ with momentum q and excitation energy $\Delta E_q = E_q - E_0$, has the following properties:

$$[H_0,B_q^+]_- = \Delta E_q B_q^+,$$

$$[B_{q_1},B_{q_2}^+]_- = \delta_{q_1q_2}, \qquad [B_{q_1},B_{q_2}]_- = 0,$$

$$B_q^+ = \sum_{P\bar{P}'}\delta_{\mathbf{p}',\mathbf{p}+\mathbf{q}}\,[G^{(+)}_{\bar{P}'\bar{P}}\gamma^+_{P'P} - G^{(-)}_{\bar{P}'\bar{P}}\gamma_{PP'}].$$

The new notation here is that $\bar{P} = (\mathbf{p}, -\lambda)$. The expansion coefficients satisfy the eigenvalue equation (for $\mathbf{p}_2' = \mathbf{p}_2 + \mathbf{q}$)

$$[\Delta E_q - (\varepsilon_{P_2'} - \varepsilon_{P_2})]\,G^{(\pm)}_{P_2'P_2} = \pm\frac{1}{\Omega} \times \sum_{P_1P_1'}(P_1'P_2|K|P_1P_2')(G^{(+)}_{P_1'P_1} + G^{(-)}_{P_1'P_1}).$$

$G^{(+)}_{P'P}$ is defined only for $p < p_0$ and $p' > |\mathbf{p} + \mathbf{q}|$, whereas $G^{(-)}_{P'P}$ is defined only for $p > p_0$ and $p' < |\mathbf{p} + \mathbf{q}|$.

The properties of the excited states have been investigated in detail by SAWADA and BROUT and others. There are, first of all, simple pair excitations with excitation energy:

$$L_{pq} = \varepsilon_{p+q} - \varepsilon_p \simeq [(\mathbf{p} + \mathbf{q})^2 - p^2]/2M^*.$$

In addition there are the genuine collective excitations whose excitation energy will be represented by ΔE_q. Some properties of these states will be discussed in detail in the next section.

Before proceeding we note that if a wave packet is formed from the collective states, the state of the system has the form

$$\psi = \exp(-iE_0t)\psi_0 + \sum_q c_q\exp(-i[E_0t + \Delta E_q]t)B_q^+\psi_0.$$

The density of particles at $(\mathbf{x},t)$ is given by the expectation value of the density operator,

$$n_\lambda = \frac{1}{\Omega}\sum_{p_1p'}a^+_{p\lambda}\,a_{p\lambda}\exp(i[\mathbf{p} - \mathbf{p}']\cdot\mathbf{x}).$$

For small deviations from the average density, this is

$$\rho_\lambda(\mathbf{x},t) = (\psi,n_\lambda\psi) \simeq \rho_0 + \rho_\lambda'(\mathbf{x},t),$$

where

$$\rho_\lambda'(\mathbf{x},t) = \sum_q C_q\exp(i[\mathbf{q}\cdot\mathbf{x} - \Delta E_qt]) \times 2\mathrm{Re}\left[\frac{1}{\Omega}\sum_p\{G^{(+)}_{p+q,-q;-\lambda-\lambda} + G^{(-)}_{p+q,-q;-\lambda-\lambda}\}\right]. \tag{8}$$

This shows that, for small amplitude oscillations, the $G^{(\pm)}$ functions are linearly related to the density fluctuations.

4. PROPERTIES OF THE COLLECTIVE MODES

In order to study the properties of these solutions it is useful to consider simple forms for the interaction K between quasiparticles. As a simple example we consider

$$(P_1'P_2'|K|P_1P_2) = \delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} \times \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_1'+\mathbf{p}_2'}f_{\lambda_1\lambda_2}v((\mathbf{p}_1 - \mathbf{p}_1')^2).$$

Because v depends only on the momentum transfer this is an ordinary local interaction. Note that it is not an exchange interaction, but simply a force where the strength depends on the spin state of the particles. We adopt the following notation for the index λ for the case of nuclear matter: $\lambda = 1,2$ corresponds to protons with spin up and spin down, respectively; $\lambda = 3,4$ corresponds to neutrons with spin up and down, respectively. The $f_{\lambda_1\lambda_2}$ then form a matrix

$$= \begin{pmatrix} c & b & a & b \\ b & c & b & a \\ a & b & c & b \\ b & a & b & c \end{pmatrix},$$

where a describes the interaction of particles with unlike spin; and b and c describe the interaction of particles with the same spin but unlike and like isotopic spin, respectively.

This model of nuclear matter might be called a 'four-fluid' model. One might expect, *a priori*, that in the normal modes of oscillation of the fluids, the density ratios would be

$$\rho_1' : \rho_2' : \rho_3' : \rho_4' = \begin{cases} 1: \ 1 \ :(-1):(-1) & \text{I} \\ 1:(-1):(-1): \ (1) & \text{II} \\ 1:(-1): \ (1) \ :(-1) & \text{III} \\ 1: \ 1 \ : \ 1 \ : \ 1 & \text{IV} \end{cases} \tag{9}$$

In the last case, the four fluids oscillate in phase, whereas in the first case the two proton fluids move out of phase with the two neutron fluids. In case III the two spin-up fluids move out of phase with the two spin-down fluids. In case II, the two fluids with opposite spin and opposite isotopic spin are out of phase. We may then give these modes the appropriate names:

$$\begin{aligned} &\text{I} && \text{Isotopic Spin Wave} \\ &\text{II} && \text{Spin-Isotopic Spin Wave} \\ &\text{III} && \text{Spin-Wave} \\ &\text{IV} && \text{Compression Wave.} \end{aligned} \tag{10}$$

Mode I is reminiscent of the Goldhaber-Teller explanation of the giant dipole resonance in the nuclear photo-effect.

The actual solution of the problem gives just these normal modes. In the long wavelength limit, i.e. $q \ll p_0$, the excitation energies are determined by the following equation for $A_q > 1$:

$$1 - \frac{A}{2} A_q \log \frac{A_q + 1}{A_q - 1} = Q^{-1}(q). \tag{11}$$

Here $A_q = M^* \Delta E_q / p_0 q$, /i.e. A_q is the excitation energy divided by the maximum pair excitation energy, and $Q(q)$ is an eigenvalue with the following values for the four normal modes:

$$Q(q) = \left[-\frac{3}{8} \frac{A}{\Omega} \frac{(Vq)}{\varepsilon_0} \right] \begin{cases} c - a & \text{I} \\ c - a & \text{II} \\ c + a - 2b & \text{III} \\ c + a + 2b & \text{IV} \end{cases} \tag{12}$$

Fig. 1 shows Q^{-1} as a function of A in the range $1 \leq A \leq \infty$. We see that Q^{-1} increases monotonically from $-\infty$ to 0 in this range. Thus if equation (12) yields a negative value for Q^{-1}, $A > 1$, and the mode will be a stable oscillation with an excitation energy determined by equation (11). On the other hand, if Q^{-1} is positive, the corresponding mode will be unstable since A will be complex.

Although carried out for a very special case, this

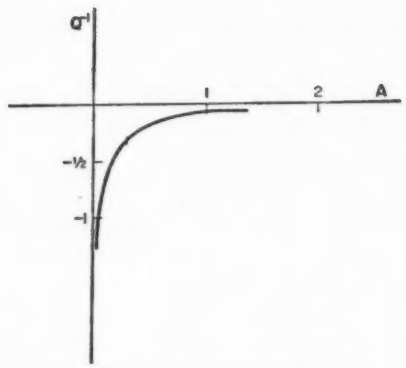


FIG. 1.—The function Q^{-1} plotted against A.

analysis has a number of interesting features. First we note that the *finite range* of the forces under consideration implies that $V(0)$ is finite. Thus for small q, the eigenvalues Q^{-1} and therefore A, are independent of q. Therefore

$$\Delta E_q = (A v_0) q, \tag{13}$$

where $v_0 = p_0 / M^*$ is the Fermi velocity. Thus the collective excitations have a phonon energy spectrum, i.e. the excitation energy is proportional to the momentum. Of course we mentioned before that the operators which create and destroy these excitations have boson commutation rules.

This result differs from that for the plasma oscillations of the electron gas and depends on the short range of the forces. On the other hand, the dispersion relation is similar to the one appropriate to the electron gas. The function Q is proportional to the Fourier transform of the potential; the increase in the number of degrees of freedom manifests itself through a different proportionality constant for each mode.

According to equation (13), A is simply the velocity of the waves in units of the Fermi velocity. In the limit of very weak forces the excitation energies and velocities of all the modes become the same and equal to the Fermi velocity:

$$A = v_0 \qquad |Q| \ll 1.$$

For 'strong' forces

$$A^2 = \left(\frac{\omega_0}{2\mu} \right)^2 \begin{cases} c - a & \text{I} \\ c - a & \text{II} \\ c + a - 2b & \text{III} \\ c + a + 2b & \text{IV} \end{cases} \qquad |Q| \gg 1.$$

In this discussion we have considered V to be a Yukawa potential of range μ^{-1} and strength g^2

$$V(q^2) = \frac{4\pi g^2}{\mu^2 + q^2}.$$

VOL 2 196

The frequency ω_0 is a characteristic frequency:

$$\omega_0^2 = \left(4\pi \frac{N}{\Omega} g^2 \frac{1}{M}\right).$$

We have also investigated the properties of the collective oscillations for the following non-local exchange interaction which is more general than the above simple example

$$(P_1'P_2'|K|P_1P_2) = v((\mathbf{p}_1 - \mathbf{p}_1')^2) \times w((\mathbf{p}_1 - \mathbf{p}_2')^2) \times (\lambda_1'\lambda_2'|1 + C_1\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + C_2\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + C_3\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2|\lambda_1\lambda_2).$$

The inclusion of the exchange mixture gives rise to 16 rather than 4 modes of oscillation, although there are only 4 distinct eigenfrequencies. If $G_{pq}^{(\pm)}$ represents the 4×4 matrix formed by the elements $G_{p'p,\lambda'\lambda}^{\pm}$ with $p' = p + q$, then the oscillating amplitudes are sums of quantities like

$$\begin{array}{lll} T[\tau_i G_{pq}^{(\pm)}] & d = 3 & \text{I} \\ T[\sigma_i\tau_j G_{pq}^{(\pm)}] & d = 9 & \text{II} \\ T[\sigma_i\tau_i G_{pq}^{(\pm)}] & d = 3 & \text{III} \\ T[G_{pq}^{(\pm)}] & d = 1 & \text{IV} \end{array} \qquad (14)$$

Here $T[\ ]$ means the diagonal sum in the λ-space, and d is the degeneracy. We have kept the same labels for the modes as in the previous example because the density ratios of equation (9) and the descriptions of equation (10) are also suitable here.

A more detailed discussion of these waves is given by GLASSGOLD, HECKROTTE and WATSON (1959). We should note that equation (13) is preserved and follows from the short-range nature of the forces. The form of the dispersion relation is modified, however, by the non-local character of the force. In addition to the function on the left side of equation (11), some of its derivatives are involved.

Even though $A_q > 1$, there is still damping from the terms V' in the original Hamiltonian left out of the discussion so far. Perturbation calculations of their effect on the lifetime of these states indicate that the condition $A_q > 1$ is actually a sufficient one for stability.

5. LANDAU'S THEORY OF THE FERMI LIQUID

One of the basic assumptions in LANDAU's theory (LANDAU, 1956) is that the classification of energy levels for a system of interacting fermions, i.e. a 'Fermi liquid,' is the same as for the ideal Fermi gas. Thus one introduces a distribution $n(\mathbf{p})$ for the occupation of these levels which is a matrix in spin space. With $d\tau = (2\pi)^{-3}\,\Omega\,d^3p$ the total number of particles N and the total momentum $\mathbf{P}$ are

$$N = T\int d\tau\, n(\mathbf{p}), \qquad (15a)$$

$$\mathbf{P} = T\int d\tau\, n(\mathbf{p})\mathbf{p}. \qquad (15b)$$

The symbol T indicates, of course, a trace in spin space. LANDAU's next essential step is to introduce first and second *functional derivatives* of the total energy E with respect to the distribution function:

$$\delta E = T\int d\tau \delta n(\mathbf{p})\varepsilon(\mathbf{p}), \qquad (16a)$$

$$\delta\varepsilon(\mathbf{p}) = T'\int d\tau f(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p}'). \qquad (16b)$$

In the Fermi gas, the particles are in states of definite momentum $\mathbf{p}$ and energy $\varepsilon^{(0)} = \mathbf{p}^2/2M$; the equilibrium distribution at temperature $kT = \beta^{-1}$ is

$$n_0 = [\exp\{\beta(\varepsilon^{(0)} - p_0^2/2M)\} + 1]^{-1}. \qquad (17)$$

In the Fermi liquid, $n(\mathbf{p})$ is the distribution of quasiparticles which obey Fermi-Dirac statistics. The energy $\varepsilon(\mathbf{p})$ of a quasiparticle with momentum $\mathbf{p}$ is given by the change in the total energy E when a single quasiparticle is added in that state. The interaction between quasiparticles is specified by the function $f(\mathbf{p},\mathbf{p}')$, and is related to the change in the energy of one quasiparticle when another is added to the system. Thus a quasiparticle's energy $\varepsilon(\mathbf{p})$ depends on the distribution of the other quasiparticles.

By varying the distribution such that $\delta N = 0$, $\delta E = 0$, and $\delta S = 0$, where the entropy is

$$S = -k\,T\int dt[n\log n + (1-n)\log(1-n)],$$

the equilibrium distribution of quasiparticles is found to be

$$n = [\exp\{\beta(\varepsilon - \mu)\} + 1]^{-1}.$$

The chemical potential μ is determined from the condition $N = T\int d\tau n$. This is formally the same as the Fermi gas except that ε and μ depend implicitly on the distribution function n.

The Fermi energy ε_0 is defined as the chemical potential at absolute zero. One also defines the velocity of the quasiparticles as $\mathbf{v} = \partial\varepsilon/\partial\mathbf{p}$ and the effective mass at the Fermi surface as $M^* = p_0/v_0$.

With these few assumptions it is possible to study the properties of the system in great detail (ABRIKOSOV and KHALATNIKOV, 1958). To a large extent the quasiparticle interaction is regarded phenomenologically subject, of course, to general invariance requirements. A correlation of this model with the observed properties of He^3 is given by ABRIKOSOV and KHALATNIKOV (1958).

One of the first results deduced by LANDAU was the

speed or ordinary sound at absolute zero, calculated from the compressibility:

$$c^2 = \frac{1}{3} v_0^2 + \frac{1}{6M} (p_0/2\pi)^3 \frac{1}{4\pi} \times \int d\Omega\, (1 - \cos\theta) TT' [f/\Omega]. \quad (18)$$

The first term is appropriate to the ideal Fermi gas; the second gives corrections from the mutual interactions between particles. Here f is the interaction between two quasiparticles on the Fermi surface and depends on the angle θ between their momenta and on their spins.*

With respect to other collective oscillations, LANDAU introduced the following transport equation:

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}} \frac{\partial \varepsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}} \frac{\partial \varepsilon}{\partial \mathbf{r}} = I(n).$$

We consider n and ε to deviate slightly from the quasiparticles ground state distribution:

$$n = n_0(p) + \delta n(\mathbf{r},\mathbf{p},t),$$
$$\varepsilon = \varepsilon^{(0)}(p) + \delta\varepsilon(\mathbf{r},\mathbf{p},t).$$

Neglecting the quasiparticle collision term $I(n)$, we obtain the following linear equation

$$\frac{\partial}{\partial t} (\delta n) + \frac{\partial}{\partial \mathbf{r}} (\delta n) \frac{\partial \varepsilon^{(0)}}{\partial \mathbf{p}} - \frac{\partial \varepsilon}{\partial \mathbf{r}} (\delta\varepsilon) \frac{\partial n_0}{\partial \mathbf{p}} = 0.$$

Assuming solutions of the form $\exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]$, we find

$$(\mathbf{q}\cdot\mathbf{v} - \omega)\, \partial n = \mathbf{q}\cdot\mathbf{v} \left(\frac{\partial n_0}{\partial \varepsilon}\right) \partial\varepsilon, \quad (19)$$

where $\mathbf{v} = \partial\varepsilon/\partial\mathbf{p}$ is the quasiparticle velocity. Because of the derivative $\partial n_0/\partial\varepsilon$ on the right side, the deviations ∂n and $\partial\varepsilon$ are restricted to the Fermi surface. By making some appropriate definitions this equation can be rewritten in a familiar form. Let $u = \omega/q$ be the wave velocity, $\eta = u/v$ the ratio of wave to quasiparticle velocity, χ the angle between two quasiparticle momenta $\mathbf{p}$ and $\mathbf{p}'$ and (θ,φ) the polar co-ordinates of $\mathbf{p}$ with respect to $\mathbf{q}$. Then we find the dispersion relation

$$v(\theta,\varphi) = \frac{\cos\theta}{\eta - \cos\theta} \frac{1}{4\pi} \int d\Omega' \nu(\theta',\varphi') F(\chi) \quad (20)$$

The new functions are

$$\nu = \int d\varepsilon n(\mathbf{p})$$

$$F = 4\pi(2\pi)^{-3} \left[p^2 \frac{dp}{d\varepsilon} T' f(\mathbf{p},\mathbf{p}') \right]_{p=p'=p_0};$$

these functions are analogous to the functions $G_{pq}^{(\pm)}$ and K introduced in previous sections. For example, if the momentum dependence of F is neglected, i.e. if F is the constant $+F_0$, we obtain the dispersion relation

$$-F_0^{-1} = 1 - \frac{n}{2} \log \frac{\eta + 1}{\eta - 1}. \quad (21)$$

This is identical to equation (1) if $-F_0$ is replaced by Q.

The similarities between LANDAU's treatment and the microscopic theory given above can be demonstrated in much greater detail. It has, in fact, been recently shown by several workers [SILIN (1957), ZURIANOV (1958), GOLDSTONE and GOTTFRIED (1959) and EHRENREICH and COHEN (1959)] that SAWADA's and LANDAU's treatments are equivalent, being different expressions of time-dependent Hartree-Foch theory.

* Near $T = 0$ the damping of ordinary sound is large because of the rarity of collisions; the only compression waves are those governed by equation (20), corresponding to Mode IV above.

REFERENCES

ABRIKOSOV A. A. and KHALATNIKOV I. M. (1958) *Progr. Sov. Phys.* **1,** 68.

BRUECKNER K. A. and GAMMEL J. L. (1958) *Phys. Rev.* **109,** 1023.

EHRENREICH H. and COHEN M. H. (1959) *Phys. Rev.* **115,** 786.

FERENTZ M., GELL-MANN M. and PINES D. (1953) *Phys. Rev.* **92,** 836.

GLASSGOLD A. E., HECKROTTE W. and WATSON K. M. (1959) *Ann. Phys.* **6,** 1.

GOLDSTONE J. and GOTTFRIED K. (1959) *Nuovo Cimento* **10,** 840.

KLIMONTOVICH IU. L. and SILIN V. P. (1952) *J. Exp. Theor. Phys. U.S.S.R.* **23,** 151.

LANDAU L. D. (1956) *J. Exp. Theor. Phys. U.S.S.R.* **30,** 1058; (1957) **32,** 59; (1959) **35,** 97. Translated in *Sov. Phys., JETP* (1957) **3,** 920; (1957) **5,** 101; (1958) **8,** 70.

SAWADA K. (1957) *Phys. Rev.* **106,** 372.

SAWADA K., BRUECKNER K. A., FUKUDA N. and BROUT R. (1957) *J. Exp. Theor. Phys. U.S.S.R.* **108,** 507 and 515. Translated in *Sov. Phys., JETP* (1958) **6,** 387.

SILIN V. P. (1952) *J. Exp. Theor. Phys. U.S.S.R.* **23,** 641.

SILIN V. P. (1957) *J. Exp. Theor. Phys. U.S.S.R.* **33,** 495.

WENTZEL G. (1957) *Phys. Rev.* **108,** 1593.

ZURIANOV P. S. (1958) *J. Exp. Theor. Phys. U.S.S.R.* **35,** 448. Translated in *Sov. Phys., JETP* (1959) **8,** 309.

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 57 to 64. Pergamon Press Ltd.

THE ORIGIN OF THE CHARACTERISTIC ELECTRON ENERGY LOSSES IN ALUMINIUM AND MAGNESIUM

C. J. POWELL*
Department of Physics, University of Western Australia, Nedlands, Western Australia

Abstract—The characteristic electron energy loss spectra of aluminium and magnesium have been measured by analysing the energy distribution of 750, 1000, 1500 and 2020 eV electrons scattered by evaporated specimens through 90°. Measurements have also been made of the loss spectra of aluminium–magnesium alloys of unknown composition using a primary electron energy of 1500 eV. It was found that the loss spectra of both elements were similar in that they were composed entirely of combinations of two elementary energy losses, 10·3 and 15·3 eV in aluminium and 7·1 and 10·6 eV in magnesium. The alloy spectra were observed to consist of combinations of a loss that varied between 10·6 and 15·3 eV in different specimens, and a low-lying loss that varied between 7·1 and 10·3 eV. From measurements of the relative positions and intensities of the two fundamental losses in each element, the observed changes in position and intensity of the low-lying loss in very thin films of aluminium and the changes in position of the two fundamental losses in the alloys, it was concluded that the larger fundamental loss in each specimen was due to plasma excitation and that the smaller loss was a lowered plasma loss of the type proposed by RITCHIE.

INTRODUCTION

MANY measurements of the characteristic energy losses of electrons in solids have been made since the pioneer experiments of RUDBERG (1930 and 1936). In 1955, MARTON, LEDER and MENDLOWITZ reviewed the instrumentation and experimental methods used by previous workers, the experimental results, and the interpretation of the results. A year later, MARTON (1956) described some of the difficulties, sources of error and discrepancies in this work. As MARTON emphasized, there are often considerable discrepancies between the results of different investigators (and even of the same investigator at different times) from supposedly the same material.

The characteristic loss spectra of aluminium and magnesium have been measured and discussed by a a large number of workers. Most authors find that the loss spectrum of aluminium is made up of combinations of a ~7 eV loss and a more prominent ~15 eV energy loss, and that the spectrum of magnesium is made up of multiples of a ~10 eV loss. Though the dominant loss peaks are sharp, the differences between the various results are often greater than the accuracy claimed for the work.

Several attempts have been made to explain the origin of the energy losses in aluminium and magnesium. CAUCHOIS (1952), WATANABE (1954 and 1956), and LEDER, MENDLOWITZ and MARTON (1956) have considered that the energy losses might be due to interband electronic transitions. These workers found that there was some degree of correlation between the characteristic energy losses and the displacements of the absorption maxima from the K and L edges (on the high-energy side) in the X-ray absorption spectra. FERRELL (1956) has pointed out, however, that the final conclusion of these authors rests on the assumption that the peaks in the absorption spectra are entirely Kronig-type structure.

A more satisfactory explanation of the loss mechanism has been proposed and discussed by PINES and BOHM (1952), PINES (1956), NOZIÈRES and PINES (1958, 1959). The electron energy losses are considered to be due to the collective excitation of the conduction electrons. In the case of a free electron gas, the fundamental energy loss is given by $\hbar\omega_p = \hbar(4\pi ne^2/m)^{\frac{1}{2}}$, where ω_p is the plasma oscillation frequency and n is the electron density. In the solid state, NOZIÈRES and PINES have shown that the electron absorption energy may differ from the free electron value on account of interband electronic transitions. Where the average interband transition frequencies are much smaller than the free electron plasma frequency (as in the non-transition metals), the observed plasma frequency is not expected to be appreciably different from the free electron value. The value of $\hbar\omega_p$ for magnesium is 10·9 eV assuming two free electrons per atom, while $\hbar\omega_p$ is equal to 15·9 eV for aluminium assuming three free electrons per atom.

The correlation between the characteristic energy losses and the displacements of the absorption maxima from the K and L X-ray absorption edges can be explained by plasma excitation. PINES has also shown

* Present address: Department of Electrical Engineering, Imperial College, London S.W.7.

that, for small scattering angles, electron energy losses by plasma excitation should predominate over energy losses due to interband transitions.

RITCHIE (1957), and later FERRELL and STERN (1958) have investigated the effect of the specimen boundaries on the nature of the plasma oscillations. RITCHIE has found that the specimen boundary would be expected to cause a decrease in intensity of the plasma energy loss and the appearance of an additional energy loss either at $\hbar\omega_p/\sqrt{2}$ if the specimen can be represented as a plane parallel-sided film, or at $\hbar\omega_p/\sqrt{3}$ if the specimen consists of spherical grains. RITCHIE considered that the low-lying losses observed in many materials (such as the ~7 eV loss in aluminium) are due to this mechanism and has calculated the probabilities of energy losses at the plasma and lowered plasma frequencies. MARTON *et al.* (1958) have measured the relative intensities of the plasma and low-lying losses in aluminium and have concluded that the low-lying loss in that element is not a lowered plasma loss.

LUPTON, FERRELL and MYERS (1958) have shown that, if the ~7 eV loss in aluminium were due to an interband transition, the transition probability would be 0·7 per cent of the plasma loss excitation probability. These authors concluded that this loss might arise in carbonaceous contamination on the specimen surface. This explanation would appear unsatisfactory, as the more intense, broad ~25 eV loss in the contamination should also be detected as a large background under the aluminium peaks. It will be shown later, however, that the mere presence of a very thin layer of contamination may affect the loss spectrum of aluminium.

Nearly all of the measurements of the characteristic loss spectra of aluminium and magnesium have been made by the energy analysis of electrons that have passed through thin specimen films of the order of a hundred Ångströms in thickness (the 'transmission' technique). Primary electron bombarding energies ranging from 2 to 100 keV have been used. The preparation of thin specimen films of the necessary purity is generally difficult. The films are often prepared by vacuum evaporation onto a suitable substrate either in the vacuum of the electron energy spectrometer or in a separate unit. In the latter case, the specimens may sometimes be detached from the substrate and are usually exposed to the atmosphere before the electron energy loss analysis. This procedure results in unknown surface oxidation and contamination, the effects of which have not been properly evaluated. Even when the specimen has been prepared in the analyser, the effect of the substrate (which is of the same order of thickness as the specimen) has not been investigated. Most authors have assumed that electron scattering by the substrate or any layer of oxide has contributed only to the general inelastically scattered electron background. Measurements of the soft X-ray absorption (TOMBOULIAN and PELL, 1951, TOWNSEND, 1953) of thin films of aluminium and magnesium on zapon substrates, however, show the existence on the high-energy side of the metal L_{23} absorption edges of extra absorption peaks which are not present in freely mounted specimens of either the metal or the zapon. This anomalous absorption was attributed to surface contamination of the metal at the metal-backing interface. It is therefore not unlikely that some of the discrepancies in the characteristic electron energy loss results of different workers might be due to insufficient knowledge of the specimen composition.

These inherent difficulties in the transmission experiments can be largely overcome by the measurement of the energy distribution of electrons scattered by the surfaces of thicker specimens (the 'reflection' technique). By successive evaporations of a sufficiently thick layer of metal onto such a surface, any oxide or contamination formed on a previous deposit will be covered and, with suitable precautions, the energy spectrum of electrons leaving the surface will be characteristic of that particular material.

The reflection technique has been used to remeasure the characteristic loss spectra of aluminium and magnesium, and to make new measurements of the loss spectra of some aluminium–magnesium alloys. In particular, the present work has been undertaken to investigate further the properties of the low-lying losses. As these might be expected to be dependent on the state of the surface, efforts have been made to prepare specimens of known volume and surface purity. The reproducibility of the results, and the manner in which the loss spectra changed under various conditions, indicate that clean metal surfaces were in fact obtained.

A disadvantage of the reflection method is that only the integrated scattering through a particular non-zero angle can be observed. For each plasma excitation, the maximum electron scattering angles for the primary electron energies used in the present experiment would be of the order of five degrees, and therefore the electrons entering the analyser must also have suffered single or multiple scattering through a fixed angle (in this case 90°). The loss spectra observed in this work must be representative of electrons that have been scattered through all possible plasma scattering angles. Further, due to the dispersion of the energy loss with scattering angle, the energy loss peaks due to plasma excitation will appear broader than in transmission experiments with near-zero scattering angles.

APPARATUS

A schematic diagram of the apparatus is shown in Fig. 1. The tantalum target support T is bombarded, through an opening in the liquid air trap C, by electrons from a hairpin tungsten filament in the Pierce-type electron gun G. The target could be heated by a tungsten filament mounted internally and the target temperature could be controlled with the aid of a chromel-alumel thermocouple junction clamped against that inside face of the target bombarded by primary electrons. Two evaporating furnaces, situated above and below the electron gun, could be used to deposit metal onto the target through the same opening in the cold-trap. This trap served to reduce the rate of carbonaceous contamination (Powell, Robins and Swan, 1958) and to act as an electrostatic shield about the target. If desired, the opening in the trap could be covered by either of two shutters H to prevent evaporated material from reaching the target during operations such as gettering. The pressure in the target chamber was normally less than 2×10^{-6} mm Hg, except during evaporations when it was less than 10^{-5} mm Hg.

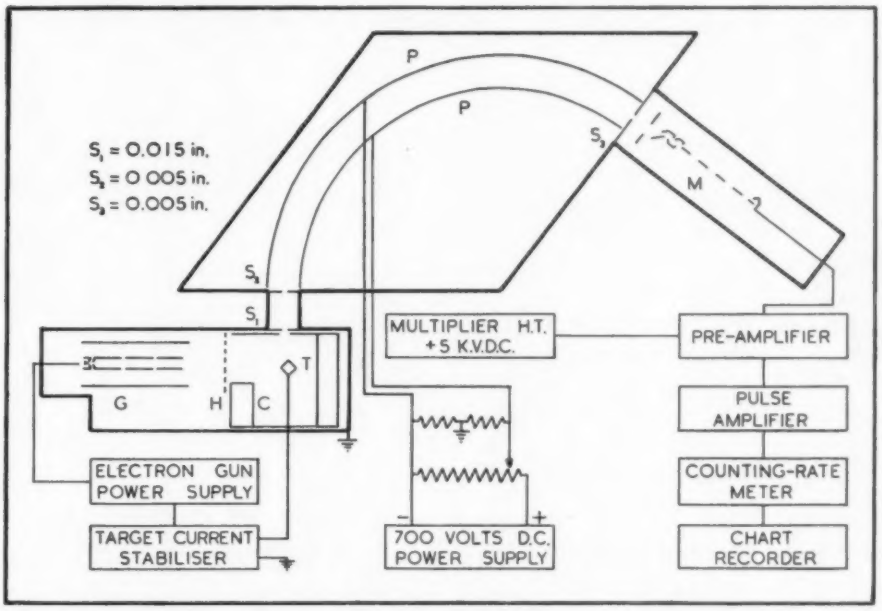


Fig. 1.—Schematic diagram of the apparatus.

Scattered and secondary electrons that passed through slits S_1 and S_2 entered the 127° electrostatic electron spectrometer (Hughes and Rojansky, 1929). The cylindrical aluminium deflector plates P, 3 in. wide and of radii 9·25 and 10·75 in., respectively, were clamped between two perspex sheets that had been coated with graphite to prevent the accumulation of a surface charge. The earth's magnetic field was reduced to less than 3×10^{-4} oersted over the region between the deflector plates using a pair of coils (Ference, Shaw and Stephenson, 1940) (not shown in Fig. 1). The axis of curvature of the deflector plates coincided with the coils axis which was aligned parallel with the direction of the earth's field.

Electrons passing through S_3 strike the first dynode of the 17-stage Be–Cu electron multiplier M. A synchronous motor was used to drive a helical potentiometer and so continuously vary the potential difference across the deflector plates, the output of the counting-rate meter being simultaneously recorded. Depending on the spectrometer resolution and the time-constant of the recording devices, the energy loss scan was generally of the order of 30 eV/min. The energy loss scale is applied by potentiometric reference to a standard cell. The present detecting system has the advantage that changes in the characteristic loss spectrum and hence in the target can be studied with respect to time.

With the slit widths shown in Fig. 1, the width at half maximum intensity of the peak of elastically scattered electrons varied between 1·35 and 2·22 eV for primary electron energies of 750 and 2020 eV, respectively. These widths include spectrometer and primary-beam energy-spread effects.

EXPERIMENTAL TECHNIQUE

All specimens were prepared by evaporation of spectroscopically pure metals. The aluminium was deposited from a multi-strand helical tungsten filament and the magnesium was deposited from a multi-strand tungsten conical basket. The aluminium–magnesium alloys were prepared by heating a helical tungsten filament on which strips of aluminium and magnesium had been placed. Though alloys of unknown composition were deposited on the target, this procedure was adopted after previous unsuccessful attempts to prepare alloys by flash evaporation of the metal filings. The nature of these alloy films will be discussed in the next section.

The evaporation time was made short in order to reduce agglomeration in the deposits, a fresh film of estimated thickness 50 to 100 Å being deposited in a few seconds. Such films were found to give loss spectra similar to those of thicker films. Most of the aluminium films were prepared, and the results taken, with a substrate temperature of about 200°C in order to further reduce the carbonaceous contamination rate. This precaution has not been necessary in subsequent work with magnesium and aluminium–magnesium alloys and the substrate was left at room temperature. It was found necessary to make several preliminary evaporations of either metal in the target

chamber, following which the specimen oxidation rate was low enough to permit a spectrum to be recorded and then repeated without any significant change taking place.

RESULTS AND DISCUSSION

(a) *Thick films of aluminium and magnesium*

The characteristic electron energy loss spectra of aluminium and magnesium have been measured for primary electron energies of about 750, 1000, 1500 and 2020 eV. The normalized spectra are shown in Figs. 2 and 3 with the intensity scales adjusted so that the peak of elastically scattered electrons has an intensity of 25 units in each case. Mean energy loss values are shown in Table 1, together with the probable error and number of measurements associated with each mean. No significant difference was found in the energy loss values for the different primary energies. It is evident that the loss spectra of aluminium are composed entirely of combinations of elementary 10·3 and 15·3 eV energy losses, while those of magnesium are composed of 7·1 and 10·6 eV energy losses. The smaller energy loss in each material is identified as the lowered plasma loss proposed by RITCHIE and the larger loss with the plasma loss observed by transmission workers.

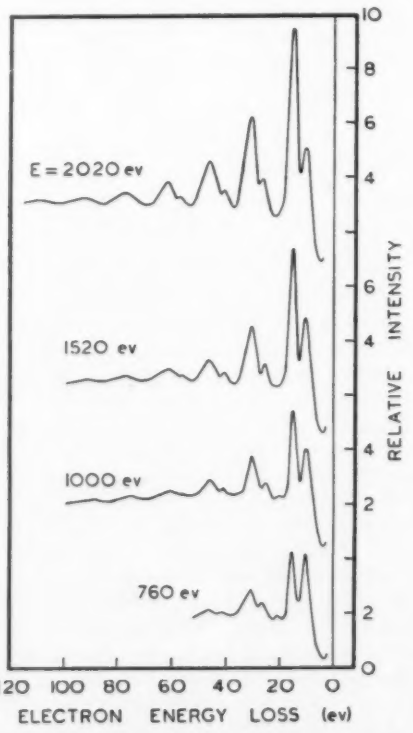


FIG. 2.—Characteristic electron energy loss spectra of aluminium for primary electron energies of 760, 1000, 1520 and 2020 eV. The peak of elastically scattered electrons (not shown) has been adjusted to an intensity of 25 units in each case.

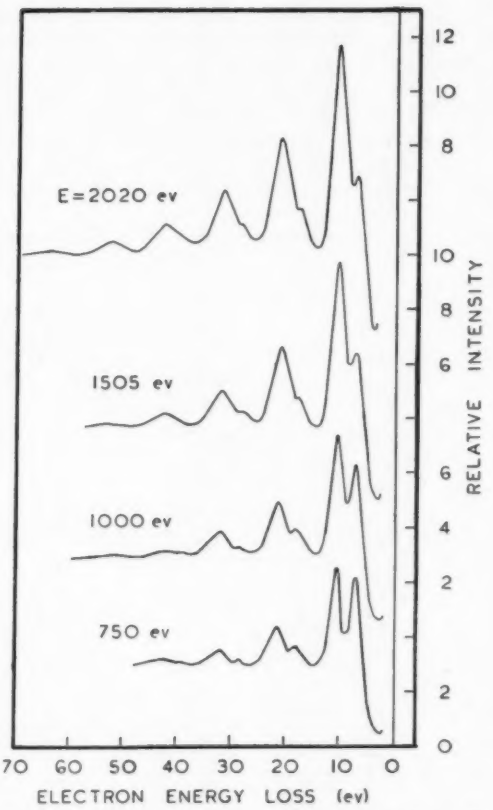


FIG. 3.—Characteristic electron energy loss spectra of magnesium for primary electron energies of 750, 1000, 1505, and 2020 eV. The intensity scales have been adjusted so that the peak of elastically scattered electrons (not shown) has an intensity of 25 units in each case.

The low-lying loss in aluminium has not been previously observed in the position quoted (10·3 eV). A low-lying loss in magnesium has only been observed by GORNYI (1958) who has recently found an energy loss of 7·5 eV with primary electron energies of 60 and 100 eV in a reflection-type experiment. It is considered that the present results are characteristic of clean metal specimens and that previous work was affected by substrate contamination or by specimen oxidation. Some effects of substrate contamination will be discussed in the next sub-section while the results of an investigation into the effects of surface oxidation will be presented in a future paper. (POWELL and SWAN, 1960).

The ratio of the fundamental energy losses in each element is $1{\cdot}49 \pm 0{\cdot}03$, slightly greater than the ratio 1·41 predicted by RITCHIE. It should be noted that the latter ratio refers to energy losses in the forward direction, whereas the measured ratio compares the mean energy losses of electrons scattered through all plasma scattering angles.

The widths at half maximum intensity of the plasma

TABLE 1.—MEASURED MEAN VALUES OF THE CHARACTERISTIC ELECTRON ENERGY LOSSES IN (a) ALUMINIUM AND (b) MAGNESIUM TOGETHER WITH THE PROBABLE ERROR AND NUMBER OF MEASUREMENTS ASSOCIATED WITH EACH MEAN.

(The symbols m and n represent the number of multiples of the low-lying and plasma lossses, respectively, that give rise to each observed energy loss).

m, n	1, 0	0, 1	2, 0	1, 1	0, 2	1, 2	0, 3	1, 3	0, 4	0, 5	0, 6	0, 7
(a) Aluminium												
Energy loss (eV)	10·3	15·3	20·5	25·6	30·5	41·1	46·1	56·0	61·4	77·0	91·8	108·6
Prob. error (eV)	0·1	0·1	0·1	0·1	0·1	0·2	0·1	0·2	0·2	0·2	0·3	0·3
Measurements	227	278	50	174	256	89	202	13	115	56	24	5
(b) Magnesium												
Energy loss (eV)	7·1	10·6		17·8	21·3	28·4	31·9	38·6	42·5	52·7	64·1	
Prob. error (eV)	0·1	0·1		0·1	0·1	0·2	0·1	0·2	0·2	0·2	0·2	
Measurements	135	145		124	142	65	140	7	111	49	12	

lines in both aluminium and magnesium have been found to be $1{\cdot}4 \pm 0{\cdot}1$ eV, after allowance had been made for the finite spectrometer resolution and the energy spread of the primary beam. This width is consistent with the measured dispersion of the energy loss and the distribution of plasma loss intensity with plasma scattering angle.

The relative intensities of the peaks in each characteristic loss spectrum and the changes in these relative intensities with varying primary electron energy have been discussed in detail elsewhere (POWELL and SWAN, 1959a,b). It might be noted here that the present measurements are consistent with the relative changes in the two fundamental losses predicted by RITCHIE.

(b) *Thin films of aluminium*

In order to examine further the properties of the two elementary energy losses, measurements have been made of the loss spectra of thin films of aluminium evaporated onto a substrate of carbonaceous contamination. Several evaporations of aluminium were made from one evaporating furnace (with the target shielded by a shutter H of Fig. 1) to getter the chamber prior to the evaporation of a known mass of commercial purity aluminium from the other furnace. The average thickness of each deposit was calculated assuming total condensation on the target and isotropic evaporation. The former assumption would appear to be reasonable in view of the results of CHANDRA and SCOTT (1958), while the latter assumption may have led to erroneous estimates of thickness. It was found, however, that the results were internally consistent and are therefore believed to be indicative of the loss behaviour for such small specimen thicknesses.

As the average film thickness was decreased, it was found that the plasma loss remained at the same position (ΔE_{pl}) while the value of the low-lying loss (ΔE_{ll}) decreased. The ratio $\Delta E_{pl}/\Delta E_{ll}$ is plotted against calculated specimen thickness in Fig. 4 for primary electron energies of 760 and 1510e V. Agglomeration would be expected to become more pronounced in the thinner films (HOLLAND, 1956) and to shift the low-lying loss in the direction observed. It was found, however, that ΔE_{ll} decreased to a value lower than $\hbar\omega_p/\sqrt{3}$, the theoretically predicted value for a film of spherical grains. At such thicknesses, the loss spectrum of the substrate predominated over that of the aluminium and it appears probable that the extra shift might be due in some manner to the substrate. Though the volume composition of the aluminium is not changed, the plasma loss being observed to remain at the same position, the aluminium-substrate interface may, in effect, contaminate the aluminium surface. It has been shown theoretically (STERN, private communication) that part of the restoring electric fields for the low-lying loss oscillations extend outside the metal, in contrast to the case of the bulk plasma oscillations where essentially none of the electric fields extend beyond the specimen. Consequently, the surface plasma oscillation is greatly affected by any films

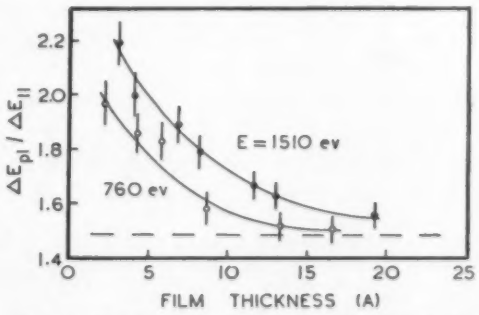


FIG. 4.—The ratio of the plasma loss (ΔE_{pl}) to the low-lying loss (ΔE_{ll}) in aluminium as a function of average film thickness for primary electron energies of 760 and 1510 eV. The dashed line corresponds to the measured value of this ratio for much thicker films.

on the specimen surface. It therefore seems likely that the effect of the substrate in the present experiment may be the cause of part of the observed shift in the low-lying loss, and may be one of the reasons why varying results are obtained with the transmission technique. It is important to note that the low-lying loss in thick, uncontaminated specimens was observed close to its predicted position.

Measurements have also been made of the plasma and low-lying loss intensities as a function of average film thickness. These measurements were subject to considerable error as it was generally difficult to locate the background which includes an unknown contribution from the substrate. The results are again consistent, however, and indicate the trend of the intensity changes. For each of the two primary electron energies concerned, the plasma and low-lying loss intensities were observed to increase nearly linearly with film thickness, approaching a maximum at the same film thicknesses as those for which the low-lying loss approached its thick-film position. These thicknesses were 15 to 20 Å and 25 to 30 Å for primary electron energies of 760 and 1510 eV, respectively. The strong non-linear increase in intensity of the plasma loss with thickness found by JULL (1956) with 10 keV primary electrons was not observed, and hence there was no need to postulate the existence of disturbed surface layers.

The change in each loss intensity with thickness was not quite linear. In fact, the ratio of the intensity of the low-lying loss to that of the plasma loss was found to increase significantly as the film thickness decreased, as predicted by RITCHIE.

The structure of the evaporated deposits is of interest in the interpretation of the results (HOLLAND, 1956). For the thicker films (50 to 100 Å) and for the rates of evaporation used, the condensed films are probably agglomerated to some extent but are definitely continuous, as no electrons scattered by the substrate are detected. For decreasing film thicknesses, the deposits would be expected to become more strongly agglomerated, but are probably continuous down to about 10 Å. It would be desirable to use primary electrons of higher energy in any similar future investigation in order that agglomeration and structure changes in the target films would be of lesser effect.

(c) *Aluminium–magnesium alloys*

RUDBERG (1936) was the first to observe that evaporated films of copper and silver, when heated to incandescence to form an alloy, gave a loss spectrum 'quite different from those for copper and for silver.' No further measurements of the characteristic energy losses of electrons in alloys have since been published. In view of the fact that sharp energy loss peaks had been observed in the loss spectra of both aluminium and magnesium and that the observed energy losses agreed closely with the free-electron predictions of the plasma theory, it was considered that it would be interesting to investigate the loss spectra of aluminium-magnesium alloys. In particular, it was thought that a plasma-type energy loss might be observed, the energy loss varying approximately as the average electron density in alloys of different compositions.

Unfortunately, it was not found possible in the present work to produce alloy specimens of known composition by vacuum evaporation. That clean unoxidized metal surfaces were essential to obtain reliable results had been shown by the work on the two elements. Though the measurements could not be related to composition, a number of significant observations were made. A primary electron energy of 1500 eV was used for all the alloy measurements.

After the simultaneous evaporation of aluminium and magnesium, a characteristic loss spectrum could be recorded with a well-defined, plasma-type energy loss between 10·6 and 15·3 eV and a low-lying loss between 7·1 and 10·3 eV. Some typical characteristic loss spectra of the alloys are shown in Fig. 5. It was found that whatever the position of the plasma loss, the low-lying loss had moved to a position such that there was no significant deviation from the relation $\Delta E_{pl}/\Delta E_{ll} = 1{\cdot}49$ found for both aluminium and magnesium. Further, the ratio of the peak intensity of the plasma loss to that of the low-lying loss (without correction for background) remained the same as that found in the elements.

It was observed that the alloy spectra did not just arise from the superposition of an aluminium and a magnesium loss spectrum. As the instrumental resolution was sufficient to separate energy loss peaks of 10·6 and 15·3 eV, and of 7·1 and 10·3 eV, it was concluded that true alloys had been formed; that is, there was effectively no separate aggregation of the elements. It would also appear from the consistency of the results that the alloys must have been reasonably homogeneous in the surface layers.

Though the general form of the characteristic loss spectra of the alloys was similar to that of the elements, it was found that the breadth of the plasma loss peak increased considerably in the alloys. The measured width of this peak at half-maximum intensity is plotted against plasma energy loss in Fig. 6(b). It is seen that the halfwidth increases continuously from 3·2 eV in magnesium to a maximum of about 5·5 eV at

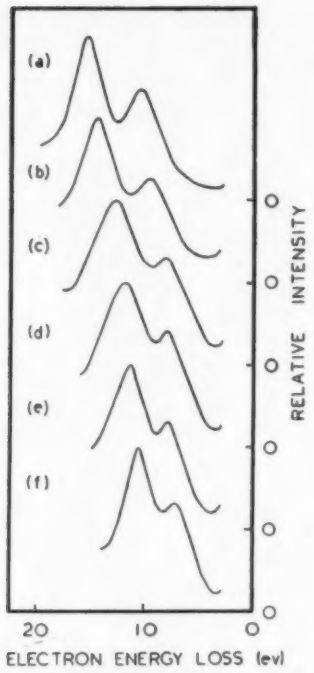


FIG. 5.—The first two energy loss peaks in the characteristic loss spectra of some aluminium-magnesium alloys, recorded with a primary electron energy of 1500 eV. Curve (a) was obtained from an aluminium specimen and curve (f) was recorded with a magnesium specimen. Curves (b) to (e) show the changes in the plasma and low-lying energy losses in the alloys. All spectra have been normalized so that the plasma loss has the same intensity in each curve. The zero of each intensity scale is indicated.

a plasma energy loss of about 12·9 eV, and then decreases to 3·2 eV for the 15·3 eV plasma loss in aluminium. As the width at half-maximum intensity of the peak of elastically scattered electrons was $1{\cdot}9_7$ eV, the true breadth of the plasma peak was increased by a maximum factor of about 2·5. The measured halfwidth of the low-lying loss is similarly plotted against the plasma energy loss position in Fig. 6(a). Only a slight gradual increase in the halfwidth of this loss is observed on going from magnesium to aluminium.

The curves in Fig. 6 show a marked difference in behaviour, and hence indicate a difference in origin of the plasma and low-lying losses. As plasma-type energy losses were observed continuously between 10·6 and 15·3 eV (Fig. 6), it seems likely that the magnitude of this energy loss varies with the average free electron density in the alloys. If the plasma-type energy loss was in fact due to an interband electronic transition, it might have been expected that the aluminium losses would appear and the magnesium losses gradually disappear in alloys of increasing aluminium content, with probably a small variation in each loss position caused by changes in the band structure. The present results therefore strongly indicate that the energy losses between 10·6 and 15·3 eV observed in aluminium-magnesium alloys are due to plasma excitation. It seems likely that the increase in breadth of the plasma-type loss is due to the fact that the conduction electrons are not moving in a uniform lattice in the alloys. The variation in the interaction between these electrons and the two types of ions in the lattice might be expected to cause an increase in breadth of this loss peak.

Apart from differences caused by the change in breadth of the plasma line and by the finite resolution of the spectrometer, it can be seen from Fig. 5 that the form of the alloy loss spectra is very similar to that of the elements. As the low-lying loss moves in position such that $\Delta E_{pl}/\Delta E_{ll} = 1{\cdot}49$ within the limits of the experimental error, it can be further concluded that the low-lying loss is a lowered plasma loss.

CONCLUSION

It has been shown that the characteristic loss spectra of aluminium and magnesium are dependent on the surface and volume purity of the specimens. Measurements utilizing the reflection technique are therefore desirable in that the targets can be made thick enough to eliminate all possible effects of the substrate.

The loss spectra of aluminium and magnesium were found to be similar in that they were composed entirely of combinations of two fundamental energy losses, 10·3 and 15·3 eV in aluminium and 7·1 and 10·6 eV in magnesium. From measurements of (a) the relative intensities and positions of the two fundamental losses in each element, (b) the observed changes in position and intensity of the low-lying loss in very thin evaporated films of aluminium and (c) the changes in position of the fundamental losses in aluminium-magnesium

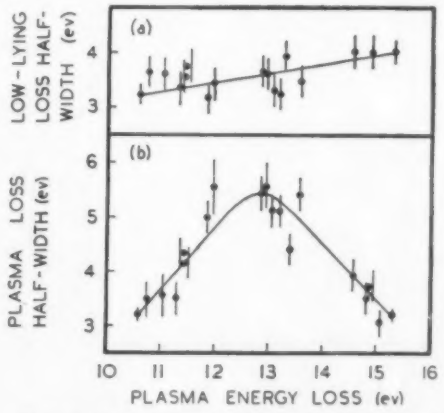


FIG. 6.—The measured widths at half-maximum intensity of (a) the low-lying loss and (b) the plasma loss plotted as a function of the corresponding plasma energy loss in aluminium-magnesium alloys. The probable error in the abscissa is ±0·2 eV.

alloys, it was concluded that the larger energy-loss component in each element was due to plasma excitation and that the smaller loss was a lowered plasma loss of the type proposed by RITCHIE.

Acknowledgements—The author wishes to thank DR. J. B. SWAN for many stimulating discussions. The electron scattering project was initiated by DR. SWAN and the apparatus was brought into its present state of operation principally through the efforts of DR. SWAN, MR. J. L. ROBINS and the author. The author finally wishes to acknowledge the financial support of the research by the Research Grants Committee of the University of Western Australia, and the tenure of the I.C.I.A.N.Z. Research Fellowship and a C.S.I.R.O. Senior Postgraduate Studentship at different times during the work.

REFERENCES

CAUCHOIS Y. (1952) *Acta Cryst.* **5,** 351.
CHANDRA S. and SCOTT G. D. (1958) *Canad. J. Phys.* **36,** 1148.
FERENCE M., SHAW A. E. and STEPHENSON R. J. (1940) *Rev. Sci. Instrum.* **11,** 57.
FERRELL R. A. (1956) *Rev. Mod. Phys.* **28,** 308.
FERRELL R. A. and STERN E. A. (1958) *Bull. Amer. Phys. Soc.* **3,** 191.
GORNYI N. B. (1958) *Izv. Akad. Nauk. SSSR (Ser. Fiz.)* **22,** 475.
HOLLAND L. (1956) *Vacuum Deposition of Thin Films* Chapman and Hall, London.
HUGHES A. L. and ROJANSKY V. (1929) *Phys. Rev.* **34,** 284.
JULL G. W. (1956) *Proc. Phys. Soc. (Lond.)* **B69,** 1237.
LEDER L. B., MENDLOWITZ H. and MARTON L. (1956) *Phys. Rev.* **101,** 1460.
LUPTON W. H., FERRELL R. A. and MYERS R. D. (1958) *Bull. Amer. Phys. Soc.* **3,** 191.
MARTON L., LEDER L. B. and MENDLOWITZ H. (1955) *Advances in Electronics and Electron Physics* **7,** 183. Academic Press, New York.
MARTON L. (1956) *Rev. Mod. Phys.* **28,** 172.
MARTON L., SIMPSON J. A., SUDDETH J. A., WAGNER M. D. and WATANABE H. (1958) *Phys. Rev.* **110,** 1057.
NOZIÈRES P. and PINES D. (1958) *Phys. Rev.* **109,** 1062.
NOZIÈRES P. and PINES D. (1959) *Phys. Rev.* **113,** 1254.
PINES D. and BOHM D. (1952) *Phys. Rev.* **85,** 338.
PINES D. (1956) *Rev. Mod. Phys.* **28,** 184.
POWELL C. J., ROBINS J. L. and SWAN J. B. (1958) *Phys. Rev.* **110,** 657.
POWELL C. J. and SWAN J. B. (1959a) *Phys. Rev.* **115,** 869; (1959b) *Phys. Rev.* **116,** 81.
POWELL C. J. and SWAN J. B. (1960) *Phys. Rev.* **118,** 640.
RITCHIE R. H. (1957) *Phys. Rev.* **106,** 874.
RUDBERG E. (1930) *Proc. Roy. Soc. (Lond.)* **A127,** 111; (1936) *Phys. Rev.* **50,** 138.
STERN E. A. Private communication.
TOMBOULIAN D. H. and PELL E. M. (1951) *Phys. Rev.* **83,** 1196.
TOWNSEND J. R. (1953) *Phys. Rev.* **92,** 556.
WATANABE H. (1954) *J. Phys. Soc. Japan* **9,** 920; (1956) *ibid.* **11,** 112.

VOL
2
196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 65 to 68. Pergamon Press Ltd. Printed in Northern Ireland

II. CLASSICAL PLASMA PHYSICS

PLASMA STUDIES SURVEY OF EXPERIMENTS—I

MAINLY OPTICAL STUDIES

K. G. EMELEUS
Queen's University, Belfast, N. Ireland

Abstract—This paper, and the succeeding one, will concentrate on certain fields in which we have experience, which seem worthy of further exploration and development. The main topics dealt with will be: the significance of gas purity and the molecular gas problem; utilization of gross unresolved appearance of a plasma and the necessity for distinguishing between steady and oscillating plasmas, illustrated by reference to terminal conditions in moving striations; noise and turbulence; a possible quantal limitation imposed on time resolution by the Einstein coefficients for allowed and forbidden transitions; formulation of the excitation integral; further examples of spectroscopic techniques, including use of interference filters, study of potential in sheaths, and a check on the voltage amplitude of so-called plasma electron oscillations.

1. INTRODUCTION

THE complete field of experimental methods for studying plasmas is too extensive for even a superficial survey in two papers. This paper, and the succeeding one, will therefore concentrate on certain fields of which we have knowledge, mainly optical in the first lecture, and mainly exploration with probes in the second. Topics omitted, or dealt with most briefly will be those which are judged to be most familiar. These include the bulk of the work on high-frequency diagnostics pioneered by Allis and Brown (see, for example, OSKAM, 1958). It is to be clearly understood however that such omissions are only made for the reasons stated, and in no sense because the work in question is unimportant.

2. GAS PURITY

In the early stages of an investigation of a plasma property, it is often unnecessary to attempt to purify rigorously the gas used. Interest frequently centres initially on gross features of the electric discharge and in some instances this may be all that it is necessary to study. Presence of impurity may even assist diagnosis, as in much of the published work on 'thermonuclear' discharges, where the main gas (ionized hydrogen or deuterium) is practically invisible and the plasma can only be seen through emission of quanta from contamination atoms, usually highly ionized.

Increasing purification, provided it does not alter the type of discharge, permits in principle of progressively more exact correlation of macroscopic properties of the plasma, and atomic properties of the substance used.

For inert gases, the methods of producing and maintaining high purity are relatively straightforward, largely as a result of the work of Penning and of Alpert, even if some details of the electrophoretic stage are not yet understood (LOEB, 1958). For molecular gases, it is doubtful if comparably high purity has ever been knowingly and unquestionably attained and maintained. Even with hydrogen made from uranium hydride, with possible low temperature fractionation, the atomic hydrogen formed in the plasma is liable to react chemically with the wall of the containing vessel. Any eventual solutions for molecular gases will probably involve continuous gas flow, and there seems room for much profitable and difficult research. In any event, it is difficult to visualize a molecular gas plasma not, in a sense, 'contaminated' by its own dissociation products.

3. GROSS APPEARANCE

It may seem unnecessary to insist on the value of monitoring a plasma by observing its gross appearance, but experience shows that this is sometimes not done carefully and significant information therefore missed. Also, there are occasions when this is impossible or difficult, for example, when the plasma is in a metal wave-guide. In these instances, it is worth considering if apparently anomalous results may not be a consequence of the plasma morphology differing from that presumed to exist.

Error can also arise from failure to recognize that apparently steady discharges may in fact contain intermittent or oscillating plasmas (cf. EMELEUS and SMITH, 1951). The techniques for establishing the

existence of such and for studying transient effects, have been developed to a high degree in connexion with 'thermonuclear' discharges. Much useful information can be obtained quickly and simply from the wave-form of the current through the plasma, by examining the light from a section of the plasma by a photomultiplier with output displayed on a c.r.o., or by a rotating mirror.

The use of the latter may be illustrated by its application to the study of moving striations. This readily reveals, for example, the fact that a large number of apparently uniform positive columns contain luminous balls moving from anode to Faraday dark space with speeds which are often of the order of the mobility speeds of positive ions (EMELEUS and OLESON, 1959), that the balls can have their origin in a growing relaxation oscillation of an anode spot (COULTER, ARMSTRONG and EMELEUS, 1958), and that an apparently stationary set of striations near the Faraday dark space may actually be a time-dependent standing wave (COOPER, COULTER and EMELEUS, 1958).

4. NOISE AND TURBULENCE

Besides oscillations with repeating wave-form, irregular oscillations or 'noise' are frequently produced by plasmas, giving rise to irregularities in current and/or voltage, and to e.m. radiation. Part of the noise, most readily studied and utilized by radiation in the microwave region of frequencies, is almost certainly largely electron-thermal, and characteristic of the electron temperature. There are perhaps difficulties in connexion with the way in which it leaves the plasma and, in view of the common occurrence of non-Maxwellian electron energy distributions, the term 'thermal' must also be looked on as inexact.

At low frequencies (say less than about 1 Mc/s) the magnitude of the fluctuations is commonly orders of magnitude above electron thermal (COBINE and GALLAGHER, 1947; MARTIN and WOODS, 1952). The noise spectrum may then be close to that of a near-random sequence of pulses of dying exponential wave-form. These can be due, for example, to instability of anode spots, or of a potential minimum in the space-charge sheath between a thermionic cathode and plasma, and can be suppressed by suitable anode construction, and by avoiding a potential minimum respectively. It is probable that there are other sources of low frequency noise, such as changes of work-function over part of a cathode due to formation or disappearance of adsorbed monolayers, and local impulsive breakdown of thicker semi-insulating patches on either anode or cathode.

The noise spectrum between about 1 Mc/s and a few hundred Mc/s is believed to be not well described or understood (MARTIN and EMELEUS, 1956). A thorough analysis on the lines developed by VAN der ZIEL (1954) might be rewarding.

The older literature contains occasional references to turbulence in plasmas and some speculations on this subject were published several years ago (EMELEUS, 1951). The subject has recently become of considerable interest, and possibly of basic significance, in connexion with 'thermonuclear' discharges

5. QUANTAL LIMITATIONS ON TIME RESOLUTION

When high-frequency plasma phenomena are studied by the light excited (whether ocularly or by photomultiplier) over increasingly short time intervals, it would appear that some failure of time resolution could result in some instances when the time-intervals in question approach the mean lifetimes of the excited atoms. From measured and computed values of lifetimes when dipole radiation is involved, this will still always permit of useful employment of high-speed photography for this purpose down to 10^{-7} sec or rather less.

The limitation might also ocasionally be severe if it became necessary to study or employ radiation from 'metastable' atoms (such as give rise to oxygen auroral radiation), or from the long lived atoms or radicles associated with active nitrogen. In such instances the times involved may approach, or exceed, a second and become greater than the characteristic diffusion time for excited atoms in the containing vessel (EMELEUS, SAYERS and BAILEY, 1950).

6. STUDY OF EXCITATION

Where no considerations like those of Section 5 arise, and fluctuations are negligible, valuable information about concentration of excited atoms can be obtained from measurement of the absolute intensities of spectra. This can be illustrated by the following somewhat idealized case.

Suppose that a small volume V of homogeneous plasma contains excited atoms of some kind with a concentration of N per unit volume, and that these emit radiation of a particular wave-length with Einstein probability coefficient A $\sec^{-1}$. Suppose further for simplicity that the radiation is emitted equally in all directions and measured with efficiency η by a recording instrument counting the quanta and subtending solid angle ω at V. Then the number of quanta recorded by the instrument per second will be

$$n = VNA\eta\omega/4\pi.$$

Using a calibrated instrument n can be determined absolutely, and hence N calculated if A is known. It is true that A has been measured directly for only a relatively small number of transitions, but it is readily calculated by wave-mechanics, provided the wave-functions are known.

The experimental techniques are straightforward (STEWART and GABATHULER, 1958) and it would appear that extensive and possibly valuable information might be readily obtained about luminous plasmas by obvious elaboration of the simple procedure just outlined.

The information so obtained is not readily accessible otherwise. It may be supplemented to a limited extent by absorption measurements, but it is to be noted that with these there is a disturbance, admittedly often small, of the pre-existing plasma which does not take place when the emitted light is utilized.

It is sometimes of value, e.g. in considering chemical reactions in plasmas, to have available an expression for the rate at which excited atoms (or molecules), of a given kind, are being produced by electron impact with the unexcited atoms (EMELEUS, LUNT and MEEK, 1936). If $Q(v)$ is the cross-section for excitation by collision of an electron with speed v with the unexcited atom, N the number of the latter per cm^3, n the number of electrons per cm^3, and $f(v)\,dv$ the fraction of the electrons with speeds between v and $v + dv$, the rate of formation of excited atoms is

$$Nn\int vf(v)Q(v)\,dv,$$

with the integral evaluated over the whole range of values of v for which neither $Q(v)$ nor $f(v)$ is zero. $Q(v)$ can be found either from low-pressure Hertz-type experiments with electrons of definite energy, or from quantum mechanics. $f(v)$ can be found from probe experiments if the pressure in the plasma is sufficiently low, but cannot at present be measured directly at higher pressure. In the latter case it is sometimes possible to obtain useful information about $f(v)$ from statistical collision theory (SMIT, 1936).

7. ELECTRON ENERGY AND POTENTIAL MEASUREMENTS FROM FIRST APPEARANCE OF SPECTRA

This again is an optical diagnostic technique which does not involve disturbance of pre-existing plasma or space-charge sheaths. It is based on the elementary fact that a least potential V^1 exists for the excitation of a specified spectrum line by electron collision with an atomic system in its ground state. Appearance of a line, provided cumulative excitation is not taking place, therefore indicates that electrons with energy of at least eV^1 are present. Potential differences or other causes giving rise to these energies are inferred.

An early example was the deduction of an excitation potential of helium by ASTON (1907) from the location of the negative edge of the first layer of light out from a cold cathode in a helium discharge, within the cathode dark space ion sheath. Reversing the line of argument, EMELEUS, BROWN and COWAN (1934) used the same effect with similar multiple layers to map roughly the potential from the cathode outward. More recently, in work not yet published in detail, LOVE (1957) has employed the same process to map the fall of potential through the space-charge heads of double striations in a hydrogen-mercury positive column.

In a different connexion, ALLEN (1955) and in work in progress here, CAIRNS and TATE, have used the appearance of argon spark lines to verify and trace the development of the Dittmer-Langmuir type of energy scattering of an electron beam by plasma-electron oscillations. Other possible applications are numerous.

8. OTHER OPTICAL TECHNIQUES

Obviously any optical technique is in principle applicable to study of a luminous plasma. Two only will be specifically mentioned here, first, utilization of polarized light and, second, the use of interference filters. An example of an elementary simultaneous application of both occurs in a paper by SHEPHERD (1959) on the polarization of H_β excited close to a cold cathode surface.

Polarization of light from a discharge is always associated with some directional effect, e.g., a magnetic field, or a directed beam of ions or electrons. Its grosser features are easy to study through the ready availability of polaroid. The main error to be guarded against is production of undesired polarization by reflection from glass or metal surfaces bounding the plasma, or in detecting instruments. It should also be borne in mind that knowledge of conditions governing production of polarized light by ion and electron impact is still incomplete (EMELEUS, 1959).

Interference filters, permitting of large aperture viewing of plasma in near-monochromatic light, have again been little exploited. They are often preferable to absorption filters, and not unduly expensive.

9. 'THERMONUCLEAR' DISCHARGES

Reference has been made to some features of these incidentally in preceding sections. The optical diagnostic methods used are described in publications from

the September 1958 Conference at Geneva on *Peaceful Uses of Atomic Energy*. With the highly ionized and violently disturbed plasmas employed, it is recognized that the utmost caution is necessary in interpretation. For example, line broadening, even if definitely due to a Doppler effect, can originate in thermal, or in mass motion of the source. Similar care in interpretation of line width is necessary with less intense plasmas. Thus an unpublished investigation by James in this laboratory of line width in some CH electronic bands from an acetylene flame would indicate a temperature of approximately 10^4 °K if interpreted as a thermal Doppler effect; actually the temperature is only of order of 4×10^3 °K, and the excess line width, as shown from the wings of the line contour, is probably due mainly to collisional broadening.

REFERENCES

ALLEN T. K. (1955) *Proc. Phys. Soc., Lond.* **68A,** 696.

ASTON F. W. (1907) *Proc. Roy. Soc.* **A80,** 45.

COBINE J. D. and GALLAGHER C. J. (1947) *J. Appl. Phys,* **18,** 110.

COOPER A. W. M., COULTER J. R. M. and EMELEUS K. G. (1958) *Nature, Lond.* **181,** 1326.

COULTER J. R. M., ARMSTRONG N. H. K. and EMELEUS K. G. (1958) *Physica, s'Grav.* **24,** 828.

EMELEUS K. G., SAYERS N. D. and BAILEY R. A. (1950) *Nature, Lond.* **166,** 656.

EMELEUS K. G. (1951) *Proc. Phys. Soc. Lond.* **64B,** 166.

EMELEUS K. G. (1959) *Proc. Phys. Soc., Lond.* **73,** 822.

EMELEUS K. G., BROWN W. L. and COWAN H. MCN. (1934) *Phil. Mag.* **17,** 146.

EMELEUS K. G., LUNT R. W. and MEEK C. A. (1936) *Trans. Faraday Soc.* **32,** 1504.

EMELEUS K. G. and OLESON N. L. (1959) *Proc. Phys. Soc., Lond.* **73,** 526.

EMELEUS K. G. and SMITH G. A. (1951) *Proc. Phys. Soc., Lond.* **64B,** 426.

LOEB L. B. (1958) *J. Appl. Phys.* **29,** 1368.

LOVE B. (1957) Thesis, Belfast.

MARTIN H. and EMELEUS K. G. (1956) *Nature, Lond.* **177,** 943.

MARTIN H. and WOODS H. A. (1952) *Proc. Phys. Soc., Lond.* **65B,** 281.

OSKAM H. J. (1958) *Philips Res. Repts.* **13,** 335 and **13,** 401.

Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva, (1958), **31, 32.** United Nations, New York.

SHEPHERD R. S. (1959) *Proc. Phys. Soc., Lond.* **73,** 126.

SMIT J. A. (1936) *Physica, s'Grav.* **3,** 543.

STEWART D. T. and GABATHULER E. (1958) *Proc. Phys. Soc., Lond.* **72,** 287.

ZIEL A. VAN der (1954) *Noise* Prentice-Hall, (and later work).

VOL
2
196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 69 to 72. Pergamon Press Ltd.

PLASMA STUDIES SURVEY OF EXPERIMENTS—II MAINLY PROBE STUDIES

K. G. EMELEUS
Queen's University, Belfast, N. Ireland

Abstract—Subjects dealt with include the classification of probes; general remarks on metal (LANGMUIR probes, and the construction of probes for volume exploration; the LANGMUIR–DRUYVESTEYN method for determining isotropic distributions, its theory, the problem of variable reflection coefficients, and BOYD's development of techniques; exploration of oscillating plasmas; effects of negative ions on plasma fields and probe currents, and on the stability of discharges.

1. CLASSIFICATION OF PROBES AND GENERAL REMARKS

USE of internal probes produces some disturbance of a plasma. This is inevitable because the volume occupied by the probe and, what is usually more significant in this connexion, its support, ceases to be available for the assembly of electrons, ions, excited atoms and quanta, which constitute the plasma. Moreover, presence of the probe may produce additional fields in the discharge, although these are often largely confined to space-charge sheaths. Much of the art of using probes consists in estimating and minimizing the extent of the disturbance they produce when they are applied in one of a number of ways.

The widest range of information is obtained with cold metal probes whose potential is deliberately varied, with simultaneous measurement of current flow between probe and plasma. This method, initiated in the now classical researches of LANGMUIR (1923) into plasmas with near-Maxwellian electron energy distributions, was first extended by LANGMUIR (1923) and his collaborators to thermionic probes and to certain forms of non-Maxwellian distribution (1926), later by DRUYVESTEYN (1930) for any isotropic electron energy distribution, and subsequently by MALTER and JOHNSON (1950) for double probes. The techniques have recently been greatly refined by BOYD (1959). The great versatility of this group of probe techniques is indicated by the fact that they have been employed in 'thermonuclear' discharges, although details of their use in this connexion are incomplete.

Electron beam probes give less information, but within their limited scope, can again be very useful. They were first employed by J. J. THOMSON and by ASTON for measuring fields, mainly in space-charge sections of glow discharges (THOMSON and THOMSON, 1933). A more refined recent application of the same technique to a similar problem is that of WARREN (1955). Good gun design may be difficult to achieve when the gun cannot be entirely shielded from the plasma. If there is much plasma in the gun, it may even lead to excitation of plasma-electron oscillations there, with consequent large energy spread in the beam, but the oscillations may be suppressed by bringing the gun-electrodes close enough together (MAHAFFEY and EMELEUS, 1958).

Electron beams are based on injection into the plasma. Withdrawal of ions from the plasma, into a magnetic mass spectrograph, as was done by LUHR (1933), or an electrostatic time-of-flight mass-spectrograph (KERR, 1956), as has been done by BOYD and by LOVE (1957), gives, conversely, information about the ion species able to reach the spectrograph's orifice from the plasma.

External metal probes have been little used and are probably worthy of more consideration. The writer's attention was first called to them in 1926 by Sir Edward Appleton. They may be used in the converse way to that in which external triggering electrodes for initiating plasmas are employed, e.g. the potential of an insulated piece of metal pressed against a glass wall will alter when an internal plasma forms, changes in composition, or is destroyed. The passage of moving striations is readily recorded and their frequency measured, by a metal loop tied round the outside of a commercial neon sign or similar tube. Suitably disposed external or internal loops may also be used to give information about magnetic fields of gaseous conductors.

Probes fail when inserted into space-charge sheaths, but the anomalous characteristics obtained may be useful for recognizing space-charges (EMELEUS and BROWN, 1936).

2. LANGMUIR PROBE CONSTRUCTION AND USE

The term Langmuir probe will be used in what follows for any metal probe inserted into a plasma, whether or not the technique employed is that originally developed by Langmuir. Opinion as to the

reliability of information obtained by a Langmuir probe varies (LOEB, 1955; EMELEUS, GREEVES and MONTGOMERY, 1936), and it is desirable to consider each individual application of such probes critically; it is doubtful if any very generally applicable critique is available.

A typical experimental problem which can arise is whether it is better to use a single movable probe for exploration of different parts of a volume of plasma, or whether it is better to use several fixed probes. The answer will naturally depend on circumstances, but it is evident that in either case a compromise is involved. A single movable probe will produce a disturbance of the plasma, which depends on its instantaneous position; a fixed probe effects analysis of a plasma disturbed permanently by the whole set of probes employed. When rigorous baking and degassing of a discharge tube is not essential, we have found a demountable probe useful, which is capable of ready movement in three dimensions by ground joints and plates. When degassing is necessary, this otherwise convenient arrangement can be partly replaced by a flexible metal bellows support for the probe lead, fused to the glass wall of the main tube containing the plasma. Alternatively, differential pumping of the region where the probe is inserted through the containing wall may be possible.

The surface of a probe is subject to bombardment by ions and electrons, and may therefore be partly sputtered or evaporated away during the course of an extended series of measurements. It is desirable to check finally whether its dimensions have become changed to an extent which, if not recognized, could lead to error in analysis of data, particularly the calculation of electron concentrations.

A subtler possible source of error can arise from alteration of the probe surface whilst a single current voltage characteristic curve is being recorded. The ion-electron bombardment of the surface will vary with the potential across the sheath and hence the adsorbed gas-content and work function of the surface may change with this potential. This could lead to a changing contact potential difference between the probe and reference electrode (usually anode or cathode) with some consequent distortion of the probe characteristic curve (VAN BERKEL, 1938; ANDERSON, 1947). This source or error might be partly avoided by making the measurements quickly, with oscillograph recording (which is, in fact, a standard and convenient technique), but again difficulty might well be encountered as the time for recording is reduced, because of the capacity and inductance (EMELEUS, 1956) of the space-charge sheaths on the probe and reference electrode. This appears to be a source of error not automatically eliminated by use of double probes.

One further source or error in the original type of Langmuir analysis may be noted here. It is well-known that there is some difficulty in locating the space-potential break on a semi-logarithmic Langmuir plot for a Maxwellian distribution. Suppose that through some misjudgment it had been taken to be ΔV negative to the true space-potential. Then the electron concentration deduced would be

$$n_0 \exp(-e\Delta V/kT),$$

where n_0 is the true concentration and T the electron temperature. The apparent depression of concentration will be, by Boltzmann's theorem, exactly that which would have existed if there had been a real drop of space-potential of ΔV in this locality in the plasma, and the potential and concentration had been measured correctly elsewhere; the error could not therefore be detected by applying this Boltzmann criterion.

3. THE LANGMUIR–DRUYVESTEYN METHOD OF ANALYSIS

In this (LD) method of analysing probe characteristics, it is not assumed *a priori* that the distribution of velocities amongst the electrons is Maxwellian, as in Langmuir's original (L) method (although this is not excluded), but merely that it is isotropic, with a distribution function $f'(v)$ in resultant velocity per unit velocity range, or $\varphi'(v)$ per unit energy range.

If i is the electron current through a probe sheath with retarding potential V to a plane, cylindrical or spherical probe with area A, then DRUYVESTEYN (1930) has shown that

$$\frac{d^2i}{dV^2} = \frac{e^2A}{4m}\frac{1}{V}f'(v).$$

This important result is actually implicit in an earlier one obtained by MOTT-SMITH and LANGMUIR. These authors shewed (LANGMUIR and MOTT-SMITH, 1926) that for the same shaped probes with the same retarding field, if one has a group of electrons with velocities of uniform magnitude, $v = \sqrt{(2eW/m)}$ and directions distributed at random, the current is a linear function of voltage, rising from zero for $v = -W$ to $ANev/4$ at the space potential, where N is the concentration of the group. The i–V characteristic has therefore slope $ANev/(4W)$. If a new group of electrons with concentration $\varphi'(W)\,dW$ starts to be received by the probe in the voltage range (W, dW), it follows that the overall slope of the i–V characteristic will change by

$$A\varphi'(W)\,dW\,e\,(2eW/m)^{\frac{1}{2}}/(4W)$$

in this range. Rearranging the notation, and changing the variable to v, yields DRUYVESTEYN's result. For many purposes $\varphi'(W)$ itself is more directly useful.

A source of error which, so far as the writer knows, has not been fully discussed, is the reflection of electrons, or emission of secondary electrons from the probe under electron bombardment. With a Maxwellian distribution in the plasma, this does not affect the temperature deduced from the probe characteristic, since the velocity distribution of electrons reaching the probe is independent of retarding voltage. For a non-Maxwellian distribution this is no longer true, and d^2i/dV^2 could depend in a complicated way on V, due to reflection and secondary emission coefficients varying with energy and angle of impact. Some uncertainty therefore arises about the accuracy in detail, although scarcely the main features, of the value of $\varphi'(W)$ derived. Non-Maxwellian distributions appear in fact to be of frequent occurrence, departure from a near-Maxwellian form being most marked for the highest energies of electron groups (BOYD and TWIDDY, 1959).

Graphical double differentiation of characteristic curves is tedious. Electronic methods for effecting this were introduced quite early (SLOANE and MACGREGOR, 1934), and have recently been brought to considerable perfection by BOYD and TWIDDY (1959). Under conditions which may be quite unfavourable on account of background discharge noise, they have found it possible to obtain details of the electron velocity distribution in striated discharges up to energies of order of 30 eV. The higher energy electrons of such groups are a kind of low voltage analogue of the runaway electrons of 'thermonuclear' discharges.

The potentialities of the LD method, with BOYD's experimental techniques, have hardly been touched. The basic theory of the method would also merit rigorous examination.

4. EXPLORATION OF OSCILLATING PLASMAS

This is another field in which further systematic investigations could be well worth while. The L and LD theories of probes were developed for steady plasmas. It is found experimentally that i–V probe characteristics taken statically in oscillating discharges often give reasonably good semi-logarithmic plots and space-potential breaks. This can occur for several reasons. For example, it may be due to the fact that a considerable stretch of many geometrical curves rising monotonically and rapidly can give a nearly straight semi-logarithmic plot; the information deduced by L or LD theory in such cases could only give at best a long-time average for the fluctuating plasma. Alternatively, as a special case, it could happen that the plasma had large density during only a small part of the oscillating discharge cycle, so that the probe characteristic would give a fairly good analysis for this part, after time scaling, if possible, for calculation of concentration.

When conditions in the plasma are fluctuating with frequencies in the kMc/S range, analysable semi-logarithmic plots may be obtained, and the electron concentration so obtained may agree astonishingly closely with those deduced from the frequencies of plasma-electron oscillations (MAHAFFEY, 1959). In this case, some simplification must result because the ions are almost unaffected by the oscillations. For fluctuations or oscillations with frequencies in the 100 kc/s–10 Mc/s range, which correspond very roughly to a commonly occurring proper frequency for ion oscillations, more complicated effects might be expected.

5. NEGATIVE IONS

The possible presence and effects of negative ions were largely ignored in early work on plasmas. An attempt to obtain information about them by probes in iodine discharges was made by SPENCER-SMITH (1935), and a preliminary theory of their effects in double space-charge sheaths between plasmas given by WITHERSPOON (1949). A general survey of their effects was given by EMELEUS and SAYERS (1938). It is now well-recognized that many atoms and radicles possess stable negative ions, and an accurate theory of probes in plasmas containing electronegative material is being developed by BOYD (1959) and his collaborators.

Negative ions cannot exist in large number in hot plasmas, because the detachment energy of the electron does not much exceed 1 eV for any negative ion and is often less. They could however conceivably play a significant part in the initial stages of such discharges and at their periphery. Also, if small fragments of the walls should break off in a hot discharge and assume a negative charge, their behaviour might simulate that of negative ions in an ordinary plasma.

Although negative ions may be formed at a small rate in plasmas, their effects may be cumulative. This is because they tend to drift towards the centre of the plasma, in the opposite direction to the positive ions. Their concentrations can then build up until they are either destroyed at the rate they are generated, by volume processes, or some new form of discharge results. If the discharge form is not altered, the plasma fields of the type that are associated with ambipolar diffusion tend to become smaller than when negative ions are absent. Of several new or modified forms which can

result when negative ions are present, one of the most interesting is a mobile thread or ribbon (SEELIGER 1932), detached from the walls, and similar in some respects to higher pressure sparks (SEYMOUR and EMELEUS, unpublished). So far as can be judged, the constriction is not due to the self-magnetic field of the discharge.

REFERENCES

ANDERSON T. A. (1947) *Phil. Mag.* **38,** 179.

BERKEL W. P. J. VAN (1938) *Physica, s'Grav.* **5,** 230.

BOYD R. L. F. and TWIDDY N. D. (1959) *Proc. Roy. Soc.* **A250,** 53.

DRUYVESTEYN M. J. (1930) *Z. Phys.* **64,** 790.

EMELEUS K. G. (1956) *Nuovo Cimento, Suppl.* **3,** 490.

EMELEUS K. G. and BROWN W. L. (1936) *Phil. Mag.* **22,** 898.

EMELEUS K. G. and SAYERS J. (1938) *Proc. Roy. Irish Acad.* **44A,** 7.

EMELEUS K. G., GREEVES F. D. and MONTGOMERY E. (1936) *Proc. Roy. Irish Acad.* **43,** 35.

JOHNSON E. O. and MALTER L. (1950) *Phys. Rev.* **80,** 58.

KERR L. W. (1956) *J. Electron. Cont.* **2,** 179.

LANGMUIR I. (1923) *J. Franklin Inst.* **196,** 751.

LANGMUIR I. and MOTT-SMITH H. M. (1926) *Phys. Rev.* **28,** 727.

LOEB L. B. (1955) *Basic Processes of Gaseous Electronics*, chap. IV, University of California Press.

LOVE B. (1957) Thesis, Belfast.

LUHR O. (1933) *Phys. Rev.* **44,** 459.

MAHAFFEY D. W. (1959) *J. Electron. Cont.* **6,** 193.

MAHAFFEY D. W. and EMELEUS K. G. (1958) *J. Electron. Cont.* **4,** 301.

SEELIGER R. (1932) *Phys. Z.* **33,** 313.

SEYMOUR R. and EMELEUS K. G. Unpublished.

SLOANE R. H. and MACGREGOR E. I. R. (1934) *Phil. Mag.* **18,** 193.

SPENCER-SMITH J. (1935) *Phil. Mag.* **19,** 806.

THOMSON J. J. and THOMSON G. P. (1933) *Conduction of Electricity Through Gases*, vol. II, Cambridge University Press.

WARREN R. (1955) *Phys. Rev.* **98,** 1650.

WITHERSPOON A. (1949) Thesis, Belfast.

VOL
2
196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 73 to 80. Pergamon Press Ltd.

CRITICAL POINTS IN THE THEORY OF ELECTRON BEAM DEVICES

J. R. PIERCE
Bell Telephone Laboratories, Inc., Murray Hill, New Jersey

Abstract—In working on the theory of electron beam devices such as travelling-wave tubes and klystrons, it has been necessary to consider a number of knotty problems, some at least of which occur in mathematical formulation of other plasma problems. Among these is the matter of multi-velocity flow. Here there are two mathematically equivalent approaches. In one, the electron flow is divided into streams according to the initial unperturbed velocities of the electrons; the variables are the densities and velocities in the various streams. In the other, velocity is regarded as a co-ordinate of phase space and the variable is the density in phase space. These approaches give different appreciations of the phenomena involved and lead to different mathematical difficulties, but are the same in content. Each demonstrates the non-existence of wave-type solutions in many infinite velocity distributions. Only a few specialized solutions of multi-velocity problems exist.

As different velocity co-ordinates can be used, so different spatial co-ordinates can be used. The Eulerian approach is common and leads to no difficulties if boundary conditions are met at fluctuating boundaries. In the case of thin beams, it is sometimes useful to use displacements from the mean position of the particles as variables. Some workers have made mistakes by mixing co-ordinate systems.

In dealing with power flow in electron beams, one can either replace the actual physical system with a linear system and find an expression for power in this system, or try to deal with the power flow in the true non-linear system. The former alternative is much simpler. Kinetic power, as well as electromagnetic power, is important in considering the orthogonality of wave-type components.

In solving an actual physical problem, one can either assemble various wave-type components so as to satisfy the boundary conditions, or solve the problem by transform or perhaps by other means.

Components of the solution must be regarded as merely a part of the solution of an actual problem, but they can sometimes be given a reasonable physical interpretation. Thus, one finds waves with positive and negative powers. When two unattenuated waves having powers of the same sign are coupled together, one observes beats. When two unattenuated waves having powers of opposite signs are coupled together, one observes growing and decaying waves. Growing waves can also be produced when a moving discontinuity couples two unattenuated waves together (parametric amplification).

THIS paper is concerned with some aspects of disturbances in clouds of charged particles. This is a particularly exasperating field of physics. What one ordinarily hopes of theoretical work is one of two things. One may hope to be able to work out an exact theory. This can be used either to predict or explain the behaviour of physical systems, or to seek for new physical effects in addition to those encompassed in the theory. Celestial mechanics approaches this ideal and so does the propagation of electromagnetic waves in passive isotropic media. On the other hand, one may be dealing with a physical phenomenon concerning which very accurate measurements can be made but concerning which one cannot work out an exact theory. In this case the validity of an approximate theory can be accurately gauged by comparing it with precise physical measurements.

I think it is no secret that it has been impossible to treat any realistic problem involving clouds of charged particles exactly. At the same time, while measurements of the behaviour of clouds of charged particles exhibit many startling phenomena, over many ranges of parameters such measurements are seldom if even highly precise. In fact, except in the case of slightly ionized plasmas they are seldom precise enough to gauge accurately the validity of various refinements or approximations made in the theory. Thus the theorist is at liberty either to make drastic simplifications or to make refined and perhaps speculative approximations, and it appears that experiments seldom catch up with him.

While measurements on electron beam devices do not have anything like the accuracy of measurements of the spectra of atoms and molecules they have perhaps been a little better than some other plasma measurements. Certainly, the experimental work has determined the need for and value of theory in connexion with such devices. It seems to me to be of some interest to consider what matters have been important in connexion with such devices and some of

the things which have been learned by studying them. In dealing with these matters I shall cite recent papers as well as original sources. I do this because a host of papers can be reached by means of the citations in later papers, while this is not true in the case of the original sources.

I think that with a few exceptions which consist of Monte-Carlo attacks on problems of high-level operation and of noise, all electron beam calculations assume a smoothed-out flow of charge rather than a cloud of particles. Thus, collisions between particles are disregarded; the charge interacts only through collective phenomena. It appears that in electron beam devices these collective interactions are much more important than interactions due to the graininess of the flow. However, one effect of this graininess must be taken into account; the noise to which it gives rise. This noise can be introduced into calculations by assigning appropriate fluctuations to the smoothed-out flow in the vicinity of the cathode.

While many useful calculations can be made in which the velocity spread of the electrons is disregarded, both experiment and theory show that this leads to errors in the computation of noise.

There are two ways of handling the problem of multi-velocity smoothed-out electron flow. In each case the equations are linearized. The two methods are equivalent in content, but the equations look somewhat different. I shall illustrate these for a one-dimensional case.

In what we may call the multi-stream approach (PIERCE, 1948) we regard the electron flow as being made up of a number of streams, and we distinguish between the unperturbed velocity u, different for every stream, and the associated a.c. or perturbation electron velocity which we call v_u. In the following equations which pertain to this case I have assumed that each quantity contains a factor

$$\exp(-j\beta z)\exp(j\omega t).$$

The charge density $\rho(u)$ of electrons in the velocity range du at u is given in terms of the total average charge density ρ_0 and a density function $\rho(u)$ by equations (1) and (2):

$$\rho(u) = \rho_0 P(u)\,du, \tag{1}$$

$$\int_{-\infty}^{\infty} P(u)\,du = 1. \tag{2}$$

We may take as the two perturbation or a–c variable quantities v_u, the a.c. velocity of electrons with an average velocity u, and i, the total convection current density. These are given in terms of the a.c. field E by (3) and (4):

$$v_u = \frac{j(e/m)E}{(\omega - \beta u)}, \tag{3}$$

$$i = j\omega(e/m)\rho_0 E \int_{-\infty}^{\infty} \frac{P(u)}{(\omega - \beta u)^2}\,du. \tag{4}$$

Here e/m, the charge-to-mass ratio of the electron, is taken as a positive quantity.

In another approach the co-ordinates used are different. The particle velocity w is regarded as a co-ordinate rather than a variable. Thus we have a density $f(t, z, w)$ which is the charge density in phase space (SIEGMANN, 1957). The equation we deal with is Liouville's theorem, which states that the density in phase space in the vicinity of any one particle does not change with time. As we disregard individual particle interactions we deal with a phase space of six dimensions. We can express the total derivative with respect to time in terms of partial derivatives taken at a fixed point, and we can express the rate of change of velocity with time in terms of the electric field. Thus, for the one-dimensional case we obtain

$$\frac{df}{dt} = 0, \tag{5}$$

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial w}\frac{dw}{dt} = 0,$$

$$\frac{\partial f}{\partial t} + w\frac{\partial f}{\partial z} = (e/m)E\frac{\partial f}{\partial w}. \tag{6}$$

These equations can be linearized by representing f as the unperturbed distribution function plus a small perturbation component, and by neglecting this perturbation component on the right-hand side of (6).

In the course of our calculations we will need various quantities such as the total charge density and the average velocity, both of which have a.c. components. These two quantities are listed in (7) and (8):

$$\rho = \int_{-\infty}^{\infty} f\,dw, \tag{7}$$

$$v = \frac{\int_{-\infty}^{\infty} wf\,dw}{\rho}. \tag{8}$$

The important application of multi-velocity flow in vacuum tubes has been, as I have noted, in the computation of noise. Here it seems necessary to resort to numerical solutions of the equations. One divides the electron flow into velocity classes, assumes a sinusoidal excitation in one class at the cathode and

computes the over all excitation at various points along the flow. This procedure is repeated with a sinusoidal excitation in each class in turn. A mean-square excitation appropriate to uncorrelated excitations in the various streams is obtained by adding the squares of the amplitudes due to the various initial sinusoidal excitations. This has been done by SIEGMAN, WATKINS and HSIEH (1957), by PIERCE and MORRISON (1958) and by CURRIE and MUELLER (to be published). In such calculations it has been found best to use the density function rather than the multistream formulation of the problem because fewer equations have to be integrated.

CURRIE and MUELLER's calculations show, for example, that the noise in electron flow accelerated by a given voltage decreases as the accelerating field is made smaller and longer; this effect is absent in single-velocity approximations. A noise lower than that predicted by single-velocity calculations is observed experimentally, and this noise is in at least rough numerical agreement with the multi-velocity computations. The multi-velocity calculations have not been carried out for the complicated geometries actually used, but for a parallel plane geometry with a specified d.c. potential.

In a particular noise problem TIEN and MOSHMAN (1956) proceeded not from the equations for smoothed-out flow, but from the dynamics of individual particles, and calculated noise by a Monte-Carlo method.

Such numerical computations are of great interest to tube designers. In many cases, however, what we would like is a little general insight into the effect of velocity distributions. To this end we can look for one-dimensional wave-like solutions in a disturbed plasma. Suppose, for instance, that we use (4) together with the law that the total a.c. current must be 0.

$$i + j\omega E = 0. \tag{9}$$

This problem was investigated by LANDAU (1946) for a Maxwellian velocity distribution and by PIERCE and MORRISON (1959) for a velocity distribution in which the charge density function is the reciprocal of a constant plus the velocity squared.

For both of these cases one indeed finds a disturbance with a decaying sinusoidal oscillating electric field. However, various components of charge density have fluctuations of other frequencies; it is only the overall charge density that has a simple exponential variation with time and distance. Thus, the disturbance is not what we usually consider to be a wave, if we say that in a wave all perturbation quantities vary in the same way with time and distance. Further, for such multi-velocity flow it can be shown that the linearized solution becomes invalid at large times even for small initial disturbances (PIERCE and MORRISON, 1959).

We can seek further insight in various ways. For instance, we can consider the behaviour of a flow consisting of a number of streams of charge densities ρ_n and velocities u_n. In this case we are faced with equation (10):

$$\frac{(e/m)}{\varepsilon} \sum_n \frac{\rho_n}{(\omega - \beta u_n)^2} = 1. \tag{10}$$

In Fig. 1, the left-hand and right-hand sides of equation (10) have been plotted as functions of the phase constant β for a case in which streams of three velocities are present. The phase constants of the waves which make up the solution correspond to the intersections of the curve for the left-hand side of the equation with the horizontal line which represents the

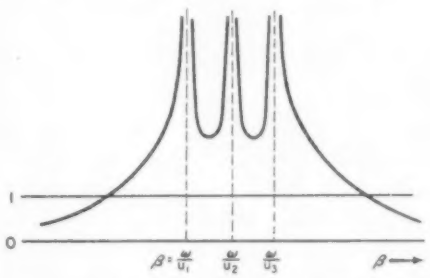


FIG. 1.—The left-hand and right-hand sides of an equation for waves in one-dimensional, three-stream flow.

right-hand side of the equation. The solutions consist of an unattenuated wave for which β is less than ω/u_1, where u_1 is the highest velocity, an unattenuated wave for which β is greater than ω/u_3, where u_3 is the lowest velocity, and two pairs of waves, each embracing a growing and a decaying wave. The rate of growth becomes very small when the velocities lie very close together. The six waves allow us to satisfy velocity and convection current boundary conditions for the three streams. The two waves whose values of β lie outside of the range of ω/u can be thought of as waves involving the whole electron flow. The other waves tend to be confined largely to particular velocity ranges of electrons, and thus the overall solution consists of two major waves plus some disturbances which we might regard as being convected with the various components of the electron flow. We should notice, however, that all of these disturbances are necessary to solve a physical problem with particular boundary conditions, such as a noise problem, and that the two unattenuated waves are not sufficient for such a purpose.

Rather than approximating the multi-velocity flow

by a number of discrete electron streams, we can expand the integrand of equation (4) in powers of $\beta u/\omega$, as in (11):

$$\frac{1}{(\omega - \beta u)^2} = \frac{1}{\omega^2(1 - \beta u/\omega)^2} = \frac{1}{\omega}(1 + 2(\beta u/\omega) + 3(\beta u/\omega)^2 + \dots) \tag{11}$$

Thus the integral becomes

$$\int_{-\infty}^{\infty} \frac{P(u)}{(\omega - \beta u)}\, du = \frac{1}{\omega^2}(1 + 2(\beta/\omega)\langle u\rangle + 3(\beta/\omega)^2\langle u^2\rangle + \dots). \tag{12}$$

If we assume the average velocity to be zero and disregard moments higher than the second, we obtain a relation which is commonly known as the dispersion relation of BOHM and GROSS (1949)

$$\omega^2 = \omega_p^2(1 + 3(\beta/\omega)^2\langle u^2\rangle). \tag{13}$$

We should note several things about this dispersion relation. If the velocity distribution is infinite, the series (11) diverges, and this is true for a Maxwellian distribution. Thus, the dispersion relation cannot be rigorously derived from the original equation. For some infinite velocity distributions it indeed does not give the same frequency of oscillation as the exact solution for the smoothed-out flow does (PIERCE and MORRISON, 1959). For the Maxwellian distribution, however, for small values of $(\beta/\omega)^2\langle u^2\rangle$, that is, for long wavelengths, the frequency is correct and the decay of the exact solution is very small. Thus the dispersion relation has some validity for the Maxwellian velocity distribution when $(\beta/\omega)^2\langle u^2\rangle$ is sufficiently small. It is dangerous, however, to apply it carelessly to large values of $(\beta/\omega)^2\langle u^2\rangle$, and it does not in any case completely describe disturbances in the plasma.

We may note that while we must make approximations which in some cases are dubious in dealing with multi-stream flow according to equations (4) and (9), these equations themselves are approximate in that they disregard the particular nature of the electron flow. Thus, seemingly important mathematical features can be concerned with velocity ranges in which the probability of finding a particle is essentially zero. What frame of mind this should leave one in I do not know. An optimist can argue that any approximation in dealing with the equations is justified in that the equations themselves are only approximate. I wish we had comparisons between predicted behaviour and actual behaviour accurate enough to test all those features of approximate solutions which are alleged to have practical importance. This seems to me to be particularly desirable when we go beyond equations in which collisions are disregarded and try to take the effect of collisions into account.

It is difficult to treat even the simplest problems of multi-velocity flow in the case of finite beams (PIERCE and WALKER, 1953 and 1956). Moreover, most of the behaviour of electron beam devices can be satisfactorily dealt with by disregarding not only the discrete nature of the electron flow but thermal velocities as well and the rest of this paper will be concerned with single-velocity flow. We have noted that in considering multi-velocity flow we have some choice as to co-ordinates, and indeed there is a wide choice of co-ordinates in dealing with single-velocity flow. These have been discussed by BOBROFF (1959). It is worth mentioning the matter because occasionally workers have mixed up co-ordinate systems and got wrong answers.

The two most common sets of co-ordinates are illustrated in the 2-dimensional case in Fig. 2. Above, we have Eulerian co-ordinates appropriate to a thick beam. These consist of values of various variables at a particular point in the flow, and the variables taken are ρ, the charge density, v_x and v_y, the velocity components of particles at a point and E_x and E_y, the a.c. electric field components. To these we may add $\tilde{y}$, the displacement of the beam boundary. This displacement $\tilde{y}$ is important because boundary conditions must be met at the wavy boundary between the beam and the charge-free space outside. However, this can be managed by assuming a surface current or surface charge proportional to $\tilde{y}$ (HAHN, 1939; RAMO, 1939; RIGROD and PIERCE, 1959).

Sometimes it is convenient to make calculations concerning a thin beam of electrons, as illustrated in the lower part of Fig. 2. Here appropriate variables are the surface charge density σ, v_x, v_y, E_x, E_y and $\tilde{y}$

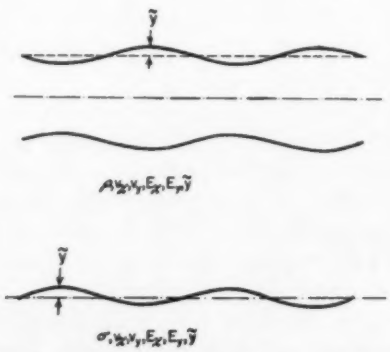


FIG. 2.—Co-ordinates for a two-dimensional thick beam and a two-dimensional thin beam.

(PIERCE, 1956). It is unnecessary to use both v_y and $\tilde{y}$, and, indeed, $\tilde{x}$ and $\tilde{y}$ can be used in place of v_x and v_y.

Power flow is a matter of primary importance in connection with disturbances on electron beams. This has been treated in a general manner by HAUS and BOBROFF (1957). Here I shall illustrate the matter of power flow in the case in which the electrons are constrained to move in the z direction only, as by an infinite magnetic field (LOUISELL and PIERCE, 1955).

Suppose we start with the general non-linear equations governing the electron stream, the force equation (14) and the definition of current density given by (15):

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{e}{m}E, \tag{14}$$

$$J = \rho v. \tag{15}$$

Using these and Maxwell's equations, we seek quantities which will satisfy equation (16):

$$\frac{\partial}{\partial z}(P_e + P_k) + \frac{\partial}{\partial t}(W_e + W_k) = 0. \tag{16}$$

Here P_e and W_e are electromagnetic power flow and stored energy density and P_k and W_k are kinetic power flow and energy density. It is clear that these quantities must be given by equations (17) through (20):

$$P_e = (\mathbf{E} \times \mathbf{H})_z, \tag{17}$$

$$W_e = \tfrac{1}{2}\mu\mathbf{H}\cdot\mathbf{H} + \tfrac{1}{2}\varepsilon\mathbf{E}\cdot\mathbf{E}, \tag{18}$$

$$P_k = -\frac{1}{2}\left(\frac{m}{e}\right)Jv^2. \tag{19}$$

$$W_k = -\frac{1}{2}\left(\frac{m}{e}\right)\rho v^2 \tag{20}$$

We note that *all* electric magnetic energies associated with the flow appear in W_e.

Almost always in vacuum tube work we deal with linearized equations. Certainly we cannot take the first-order approximations obtained through the solutions of linearized equations and safely use them in the exact expressions for a.c. power, which contain second order terms. What are we to do?

Two courses are open to us. One is to carry along higher-order terms in our solutions of the true non-linear equations. WALKER has done this difficult task (WALKER, 1954, 1955). Another simpler and satisfactory approach is to linearize the problem, not the solution (LOUISELL and PIERCE, 1955). Thus, we regard the linearized force and current equations.

$$\frac{\partial v}{\partial t} + u_0\frac{\partial v}{\partial z} = -\frac{e}{m}E_z, \tag{21}$$

$$\left.\begin{aligned} J &= \rho u_0 + \rho_0 v, \\ J_0 &= \rho_0 u_0, \end{aligned}\right\} \tag{22}$$

as the equations of a system which we are considering, a system whose behaviour will correspond closely to the behaviour of the actual non-linear system at low signal levels. We then seek to find for this linear system definitions of P_k and W_k which will satisfy (16), Maxwell's equations and equations (21) and (22). A little diligent search leads us to expressions (23) and (24) for P_k and W_k:

$$P_k = -\frac{1}{2}\left(\frac{m}{e}\right)(J_0 + J)(u_0^2 + 2u_0 v), \tag{23}$$

$$W_k = \frac{1}{2}\left(\frac{m}{e}\right)[\rho_0(u_0^2 + 2u_0 v + v^2) + \rho(u_0^2 + 2u_0 v)]. \tag{24}$$

We should note that these do not contain all possible second-order terms.

For sinusoidal excitations the a. c. power density may be written simply in terms of the convection current density J and a kinetic voltage U, as indicated in (25) and (26):

$$P_k = \tfrac{1}{2}(JU^* + J^*U), \tag{25}$$

$$U = \frac{-u_0 v}{(e/m)}. \tag{26}$$

The presence of kinetic power as well as of electromagnetic power is important in connexion with the orthogonality of modes of propagation in an electron stream. For purely electromagnetic waves the condition of orthogonality is that the power in the direction of propagation computed by using the electric field of one mode and the magnetic field of another mode is zero. However, orthogonality must involve total power, electromagnet and kicinetic. Thus the orthogonality relation is

$$0 = \int[(\mathbf{E}_n \times \mathbf{H}_m^* + \mathbf{E}_n^* \times \mathbf{H}_m) + (\mathbf{U}_n\cdot\mathbf{J}_m^* + \mathbf{U}_n^*\cdot\mathbf{J}_m)]\,ds, \qquad m \neq n \tag{27}$$

Here the field components are normal to the direction of propagation and $\mathbf{J}$ and $\mathbf{U}$ are along the direction of propagation; the integral is over the area of the wave. BRESSLER, JOSHI and MARCUVITZ (1958) have dealt with this matter.

Ordinarily it is most convenient to think of the excitation of an electron stream in terms of modes of propagation, but these are after all only the components of a solution to an actual physical problem. One can solve the problem either by finding all the relevant modes and fitting the boundary conditions, or he can use some sort of transform method (RIGROD and PIERCE, 1959; WALKER, 1954). Either in the search for the modes or in the solution of problems by

transform methods, admittance and impedance can be very useful concepts. Consider, for example, an axially symmetrical disturbance on a cylindrical electron stream with some specified phase constant and frequency. At some point outside the stream there will be a certain circumferential magnetic field and an axial electric field. At any radius we can regard the ratio of H_θ to E_z as an admittance. In some cases there may be another component of admittance involving E_θ and H_z. H_θ also gives the total current I_z within the radius R. Thus the admittance Y is

$$Y = \frac{H_\theta}{E_z} = \frac{I_z}{E_z}. \tag{28}$$

Once we have the admittance Y, we can consider the problem of the electron stream surrounded by a conducting tube at the radius R. We can, for instance, specify E_z as zero except across an infinitesimal gap at $z = 0$ and thus find the current everywhere in the wall of the tube resulting from excitation of the beam by a voltage across a gap in the tube (RIGROD and PIERCE, 1959).

We can also use Y in finding the normal modes. The rule is that for a normal mode the sum of the admittances looking in toward the beam and out away from the beam must be zero,

$$Y_{\text{in}} + Y_{\text{out}} = 0 \tag{29}$$

The admittance looking outward into a conducting tube is infinite and thus the normal modes for a beam inside a conducting tube at radius R occur at the poles of the admittance looking inward.

The admittance Y looking in toward a thick electron beam usually has a large number of poles for values of ω and β such that the frequency ω_m seen moving with the electrons is nearly equal to the frequency ω_p.

$$\omega_m = |\omega - \beta u_0|, \tag{30}$$

$$\omega_p = \sqrt{\frac{(e/m)\rho_0}{\varepsilon}}. \tag{31}$$

There are also a large number of poles when ω_m is near to the cyclotron frequency ω_c,

$$\omega_c = (e/m)B. \tag{32}$$

The different roots in these vicinities correspond to different variations of field quantities with radius. These modes have been studied in considerable detail by HAHN (1939) and RAMO (1939).

Let us for the moment disregard the effects of the cyclotron frequency and consider the two space-charge waves which correspond to the simplest and least rapid field variation with radius. One of these will be a wave whose phase velocity is faster than the electron velocity and the other will be a wave whose phase velocity is slower than the electron velocity. The fast wave has positive energy and positive power. The slow wave has a negative energy and a negative power.

How does this come to be? We can understand by considering the behaviour of the electrons in a coordinate system which moves with the phase velocity of the wave. In this case we will, in effect, see the electrons moving against and with the electric field much as balls might roll over a sinusoidal hill, as shown in Fig. 3. When the phase velocity of the wave is less than the velocity of the electrons the balls will be rolling to the right, in the direction of wave motion. They will be bunched together at the tops of the hills, where their velocity with respect to the wave is least. The velocity

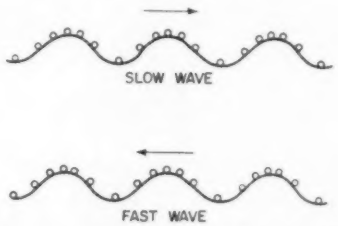


FIG. 3.—Representation of motion of electrons in the field of a slow (above) and a fast (below) wave, as seen moving with the phase velocity of the wave.

with respect to a fixed frame of reference is the velocity of the wave plus the velocity of the electrons with respect to the wave. Thus, the electrons are bunched together in positions at which their total velocity with respect to the fixed frame of reference is least. This means that in order to set up such a disturbance, energy must be abstracted from the wave, and so we must assign a negative kinetic energy and a negative kinetic power to the wave.

As shown also in Fig. 3, when we move with the fast wave we see the electrons as moving backward, to the left. Again they are bunched in regions of least velocity with respect to the wave. In this case, however, the velocity of the electrons in the fixed frame of reference is the velocity of the wave minus the velocity of the electrons with respect to the wave. Thus, the velocity in the fixed frame is greatest where the velocity of the electrons with respect to the wave is least, that is, where the electrons are bunched together. For this reason the fast wave has positive kinetic energy and positive kinetic power.

Both the slow and the fast waves have some electromagnetic power as well as their kinetic power, but for space-charge waves whose velocity is slow compared with the velocity of light the kinetic power predominates.

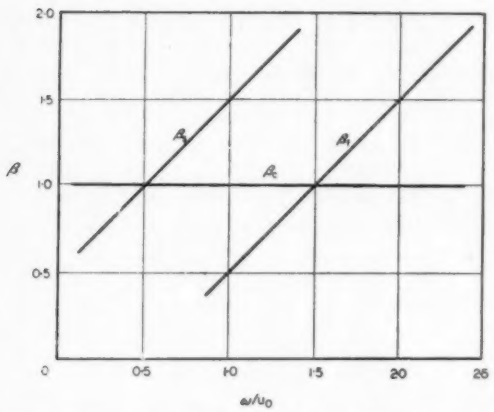


FIG. 4.—Phase constants of circuit and space-charge waves plotted against a parameter involving electron velocity u_0.

The performance of many types of electron tubes can be explained in terms of the energy of space-charge waves. Interesting phenomena in tubes can be explained in terms of the coupling of modes of propagation (PIERCE, 1954). The effect of such coupling depends on whether the signs of the powers of the waves which are coupled are the same or are different.

To illustrate this, let us consider the case of an electron beam coupled to a helix, as in the travelling-wave tube. We will seek the behaviour of the modes as we vary the velocity u_0 of the electrons. In Fig. 4 the phase constant β for various waves is plotted vertically; horizontally there is plotted a parameter ω/u_0, which changes as the electron velocity u_0 is changed. β_c is the phase constant of waves on the circuit which, of course, does not change as the electron velocity is changed. β_s is the phase constant of the slow space-charge wave on the electron stream, and β_s is the phase

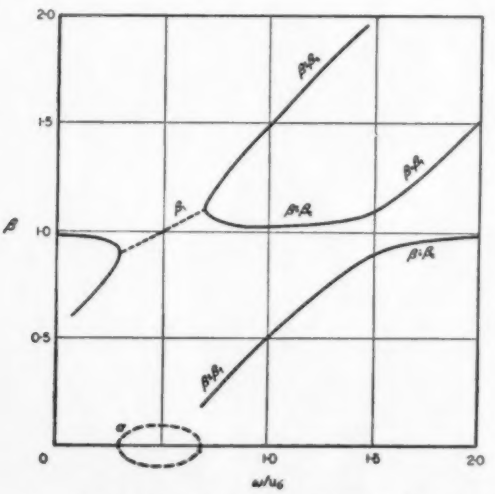


FIG. 5.—Result of coupling modes of Fig. 4.

FIG. 6.—Reflection of a wave of power P_1 from an advancing piston.

constant of the fast space-charge wave on the electron stream. We may expect some interaction for values of u_0 which make β_s or β_f close to β_c.

The interaction results in the variation of phase constant with electron velocity shown in Fig. 5. When the powers are different in sign, as in the case of the circuit wave and the slow space-charge wave, the coupling results in a pair of waves with the same phase constant, one increasing exponentially with distance and the other decaying exponentially with distance. Each is made up of a component of circuit wave and a component of slow space charge carrying equal and opposite powers, so that the total power of the decreasing wave is zero and the total power of the increasing wave is zero. The gain of the increasing wave represents, not a gain in net power, but a gain in the power of the positive-power component wave at the expense of an increasing negative power in the negative-power components.

When the powers of the two waves are both negative or both positive, as in the case of the circuit wave and the fast space-charge wave, the coupling merely results in two unattenuated waves, as shown in Fig. 5.

One other mechanism of interaction has become important in electron tubes. Consider the case of a wave of power P_1 reflected by a piston which advances with a velocity v and exerts a force f, as shown in Fig. 6. It is clear that the reflected power P_2 must be greater than P_1 and must consist of a wave of a higher frequency. Reflection from a single piston is not a stationary process, but reflection from a multitude of discontinuities, such as those caused by bunches in an electron stream, can be a continuous process, and such *parametric amplification* can be attained in various forms of vacuum tubes as well as in other electromagnetic devices (PIERCE, 1959).

REFERENCES

BOBROFF D. L. (1959) *IRE Trans.* **ED-6**, 1, 68.
BOHM D. and GROSS E. P. (1949) *Phys. Rev.* **75**, 1851.
BRESSLER A. D., JOSHI G. H. and MARCUVITZ N. (1958) *J. Appl. Phys.* **29**, 794.
CURRIE M. R. and MUELLER W. M. *J. Appl. Phys.* To be published.
HAHN W. C. (1939) *Gen. Elect. Rev.* **42**, 258.
HAUS H. A. and BOBROFF D. L. (1957) *J. Appl. Phys.* **28**, 694.
LANDAU L. (1946) *J. Phys., U.S.S.R.* **10**, 25.

LOUISELL W. H. and PIERCE J. R. (1955) *Proc. Inst. Radio Engrs. N.Y.* **43,** 425.
PIERCE J. R. (1948) *J. Appl. Phys.* **19,** 231.
PIERCE J. R. (1954) *Bell. Syst. Tech. J.* **33,** 1343.
PIERCE J. R. (1956) *IRE Trans.* **ED-3,** 183.
PIERCE J. R. (1958) *Proc. Inst. Elect. Engrs.* **105,** Part B Suppl. No. 11, 786.
PIERCE J. R. (1959) *J. Appl. Phys.*
PIERCE J. R. and MORRISON J. A. (1959) *IRE Trans.* **ED-6,** 2, 231.
PIERCE J. R. and WALKER L. R. (1953) *J. Appl. Phys.* **24,** 1328.
PIERCE J. R. and WALKER L. R. (1956) *Bell. Syst. Tech. J.* **35,** 109.
RAMO S. (1939) *Phys. Rev.* **56,** 276.
RIGROD W. W. and PIERCE J. R. (1959) *Bell Syst. Tech. J.* **38,** 99.
SIEGMAN A. E. (1957) *J. Appl. Phys.* **28,** 1132.
SIEGMAN A. E., WATKINS D. A. and HSUNG-CHENG HSIEH (1957) *J. Appl. Phys.* **28,** 1138.
TIEN P. K. and MOSHMAN J. (1956) *J. Appl. Phys.* **27,** 1067.
WALKER L. R. (1954) *J. Appl. Phys.* **25,** 615; (1955) *J. Appl. Phys.* **26,** 1031.

VOL.
2
196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 81 to 83. Pergamon Press Ltd. Printed in Northern Ireland

PLASMA DIFFUSION IN STELLARATORS*

T. COOR
Project Matterhorn, Princeton University, Princeton, N.J.

Abstract—During ohmic heating of plasma confined in a Stellarator it is observed that the plasma moves across the magnetic confining field at a much higher average velocity than is predicted by classical considerations. An attempt has been made to compare the observed velocity with the calculated drifts that should result from the measured randomly varying electric fields and density gradients present in the plasma. It is found that the random quantities, as measured by double probes, are of sufficient intensity to give order-of-magnitude agreement between observed and calculated velocities. A new hypothesis is put forward to explain the origin of these high-amplitude fluctuations, a hypothesis which might be applicable to gas discharges generally. It is assumed that the non-Maxwellian electron distribution resulting from the applied electric field excites large amplitude, coherent plasma oscillations near the plasma frequency. These large amplitude oscillations act as the 'pump' in the non-linear process known as parametric amplification. Thermal fluctuations, normally present with small amplitudes in quiescent plasmas, are amplified over a wide band of frequencies, both above and below the pump frequency, by the resulting parametric amplifier action of the plasma. This results in high-amplitude noise which might produce enhanced diffusion and rapid thermalization effects.

EXTENSIVE calculations and experiments have been carried out in the last few years on the ohmic heating and confinement of plasma in the Stellarator.† Theory indicates that apart from possible electrostatic or hydromagnetic instabilities, plasma confinement in magnetic geometries possessing rotational transform should be governed by classical diffusion, that is, diffusion across the field arising from collisions between particles. The hydromagnetic instabilities that might be expected to affect confinement fall into two catagories: (i) those arising when the plasma filament carries a net axial current of a certain critical amount (KRUSKAL, JOHNSON, GOTTLIEB and GOLDMAN, 1958); and (ii) those that occur at large plasma pressures (relative to magnetic pressure) and arise from the diamagnetic nature of plasma.† The instability associated with axial current, which is known as the KRUSKAL instability, has been observed in some detail during the operation of several Stellarators at Princeton (KRUSKAL, JOHNSON, GOTTLIEB and GOLDMAN, 1958). As yet, there have been no observations of the interchange instability, as those of the second catagory are called. The presence of one or more of these hydromagnetic instabilities would be expected to move plasma to the walls of the Stellarator with much higher velocities than would classical diffusion. The same might be true if large random or oscillatory electric fields exist, arising from co-operative phenomena excited by the externally applied ohmic heating field. Diffusion having its origin in co-operative phenomena is the subject of this paper.

Among the main results obtained in the operation of Stellarators has been the observation that during ohmic heating the plasma is confined for times three orders of magnitude shorter than predicted from classical diffusion (COOR, CUNNINGHAM, ELLIS, HEALD and KRANZ, 1958). It has not been possible to study the confinement of plasma in the quiescent state, after termination of the ohmic heating pulse, because recombination stimulated by discharge-liberated impurities is the dominant process. An explanation of the rapid diffusion during heating and a determination of confinement of a hot, quiescent plasma remain major problems of Stellarator physics.

Recent experiments on plasma heating and confinement in the B-3 Stellarator at Princeton (ELLIS, GOLDBERG and GORMAN, to be published) have made some important contributions to our understanding of the nature of the process giving rise to enhanced diffusion. Here a special stabilizing field configuration that theoretically mitigates hydromagnetic instabilities of both types was added to the figure-eight geometry. It was found that the use of this stabilizing field did not materially affect the rate of plasma loss from the discharge. On the other hand, these experiments indicate that the onset of rapid diffusion occurs when the ratio of energy gained by electrons in one mean free path to average electron energy reaches the critical value of about 0·1. These results strongly

* Supported by the U.S. Atomic Energy Commission under contract AT(30–1)–1238 with Princeton University, Princeton, New Jersey.

† A complete survey of the Stellarator concept is given by SPITZER (1958).

indicate that some type of instability associated with 'runaway' electrons (BERNSTEIN, CHEN, HEALD and KRANZ, 1958) or skew electrons distributions might be responsible rather than hydromagnetic processes.

In order to study the relation between enhanced diffusion and co-operative phenomena in Stellarator plasma a series of experiments have been carried out on *Etude*, (JONES, UPHAM and COOR, to be published), a low-field, high-pulse-rate Stellarator which utilizes an axial field plus helical stabilizing field in a toroidal discharge tube for confinement. Characteristic comment times for hydrogen in Etude are of the order of 100 μsec in its 5 cm diameter discharge tube, giving a mean plasma velocity to the walls of $V_D \cong 2 \times 10^4$ cm/sec. Floating and ion-saturated double probe measurements have been made with the following general results:

(1) The floating double probes indicate a randomly varying electric field with an RMS value of the order of 1 volt/cm across the magnetic field. The frequency (power) spectrum of this noise is constant from 50 kc (the lowest frequency that can be measured up to 200 kc from whence it falls off at about 9 dB per octave out to 30 Mc, the highest frequency measured.

(2) The scale distance for the disturbance, as determined by varying the probe separation, is inversely related to frequency and is characteristically about 3m at 100 kc.

(3) Ion-saturated double probe measurements indicate high-frequency ion density fluctuations across the field of almost 100 per cent, i.e. $\Delta n_i/n_i \approx 1$.

To see if these results are consistent with the observed plasma behaviour let us look at the important terms in the basic equations of motion of the plasma.* They can be written as

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} - \frac{c}{en_e} \Delta p_i = 0. \tag{1}$$

Solving for the velocity perpendicular to the magnetic field and substituting $kT_i \, \Delta n_i = \Delta p_i$, we have

$$\mathbf{V}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{ck}{e} T_i \frac{\Delta n_i \times \mathbf{B}}{n_i B^2}. \tag{2}$$

Thus fluctuating electric fields or ion density gradients occurring normal to the magnetic field can cause plasma motions *across* the field. Unfortunately, it is not possible to simultaneously measure $\mathbf{E}$ and Δn_i at a point in the plasma, as would be necessary to evaluate the random walk of the plasma under the influence of the two driving terms which are of the same magnitude in Etude. Under the rather dubious assumption that the two effects act independently on the plasma, it has been shown (JONES, UPHAM and COOR, to be published) that either the electric field or pressure gradient fluctuation is of sufficient magnitude to move the plasma across the discharge tube in 100 μsec.

The question arises as to the origin of these fluctuations. During ohmic heating the applied electric field certainly distorts the electrons away from a normal distribution and there is much evidence for the excitation of co-operative phenomena (BERNSTEIN, CHEN, HEALD and KRANZ, 1958). There are many suggestions as to how a skew electron distribution or runaway electrons can excite various instabilities. However, most of the disturbances that can be shown to arise are near the electron or ion plasma frequency and hence much too high (kilomegacycles) to be directly responsible for rapid diffusion. One type of process that can give rise to lower frequency disturbances was suggested by the author and investigated in detail by DAWSON and BERNSTEIN (1958). Here high velocity electrons moving along lines of force can interact with Alfven waves and transfer energy to them. This occurs when the Larmor radius of the electron becomes comparable with the wavelength of the disturbance, or when the electron sees its own cyclotron frequency in traversing the disturbance. But again the frequency of such phenomena are too high to be effective in producing enhanced diffusion.

A hypothesis that might explain the high amplitude, broad-band noise observed in Stellarators and in gas discharges generally is to be found in the non-linear process known as parametric amplification. The non-Maxwellian electron distributions resulting from the applied electric fields can, in general, excite large amplitude coherent oscillations near the plasma frequency. These high frequency disturbances will, if of large enough amplitude, drive the plasma into the non-linear limit where a large body of new phenomena arise. Among the more important consequences of this will be the following: (i) Thermal fluctuations appearing as plasma density variations, Alfvén waves, sound waves, etc. will 'be pumped' by the large amplitude plasma oscillations and will gain energy in excess of the normal kT value. This occurs by virtue of coupling between modes and even between types of waves by the non-linear pumping. The energy being fed into the plasma at the primary frequency will then be distributed among various types of wave motion into a broad band of frequencies both above and below the primary, but the energy spectrum will still be peaked in the high frequency region. However, due to the non-linear mixing beats between adjacent components of the

* The basic equations are derived by SPITZER (1956).

large amplitude high frequency noise can cause additional spreading of the energy into the low frequency region. This can be looked upon as a sort of plasma 'turbulence,' where large amplitude disturbances of one type are excited at one frequency by the driving process. The energy then spreads by means of the resultant non-linear effects to all allowable modes. The turbulent fields resulting from the large amplitude low frequency noise could then random walk the plasma across the magnetic field and result in enhanced diffusion. (ii) As pointed out by HAUS (1959), large amplitude pumping of the plasma (he considers externally applied R.F. fields) will cause coupling of the modes of the plasma in such a way that energy can flow between modes. A result could be that confinement-destroying instabilities might develop. As HAUS points out, R.F. confinement schemes would be particularly subject to such effects. However, R.F. heating schemes such as magnetic pumping, ion cyclotron heating or even ohmic heating when plasma oscillations are excited might also stimulate the development of such instabilities. (iii) The high amplitude turbulence produced by non-linear parametric effects might also be responsible for the anomalously rapid thermalization of electrons and ions often observed in gas discharges.

As direct evidence that these processes are responsible for the behaviour of our devices little can be cited other than the fact that very intense disturbances are observed to be excited in the microwave region in our devices (BERNSTEIN, CHEN, HEALD and KRANZ, 1958). However, in microwave electronics there is much work in progress on parametric amplification in much simpler systems. These might furnish important analogies to plasma. A few published results on electron beam devices, non-linear transmission lines, etc. exhibit properties that are strongly suggestive of plasma and may be especially relevant. RIGROD (1957) reports on experiments where thermal noise in a rippled-beam tube is amplified by some non-linear process, presumably parametric amplification, excited by pumping oscillations some decade in frequency above the noise. ASHKIN (1958) reports the parametric amplification of coherent signals superimposed on high-amplitude velocity modulated electron beams. These two results indicate that if two signals of different frequency, one of large amplitude and the other of small, propagate through a non-linear, non-resonant medium together there is, apart from the sum and difference beats that are normally generated, amplification of the weak signal at the expense of the large.

It is the author's belief that much of the anomalous behaviour of gas discharge plasma is determined by non-linear processes and that a serious attempt should be made to study non-linear phenomena, both theoretically and experimentally. Even if only approximate theories or simple analogue simulators can be studied at this time it is possible that important insight can be gained by this attempt.

REFERENCES

ASHKIN (1958) *J. Appl. Phys.* **24,** 1646.

BERNSTEIN W., CHEN F. F., HEALD M. A. and KRANZ A. Z. (1958) *Phys. Fluids* **1,** 430.

COOR T., CUNNINGHAM S. P., ELLIS R. A., HEALD M. A. and KRANZ A. Z. (1958) *Phys. Fluids* **1,** 411.

DAWSON and BERNSTEIN (1958) *Controlled Thermonuclear Conference,* Washington D.C., TID-7558.

ELLIS R. A., GOLDBERG and GORMAN, *Phys. Fluids.* To be published.

HAUS H. A. (1959) M.I.T. Research Laboratory of Electronics, Quarterly Progress Report No. 52 (January)

JONES UPHAM and COOR. To be published.

KRANZ A. Z. (1958) *Phys. Fluids* **1,** 411.

KRUSKAL M. D., JOHNSON J. L., GOTTLIEB M. B. and GOLDMAN L. M. (1958) *Phys. Fluids* **1,** 421.

RIGROD (1957) *Bell Syst. Tech. J.* (1957) **36,** 831.

SPITZER L. (1956) *Physics of Fully Ionized Gases,* Chap. 2, Interscience, New York.

SPITZER L., JR. (1958) *Phys. Fluids* **1,** 253.

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 84 to 87. Pergamon Press Ltd. Printed in Northern Ireland

ION CYCLOTRON HEATING OF A PLASMA

T. H. STIX
Project Matterhorn, Princeton University, Princeton, N.J.

Abstract—Experiments with ion cyclotron heating of a deuterium plasma confined in the B-65 stellarator are reported. Single-particle type behaviour is evidenced by the production of neutrons, ascribed to deuterons accelerated in the low density region beneath the induction coil but at the outside of the plasma, at a value B_0 of confining magnetic field corresponding to single particle resonance. Ion cyclotron wave behaviour is indicated by the emission of light from deuterium and from several times ionized impurity elements, the light intensity peaking at a value of the magnetic field about 10 per cent larger than B_0. Maximum absorption of the ion cyclotron wave energy is calculated to occur in a cylindrical shell where the ion density is about 10^{13} cm^{-3}, and qualitative support is given experimentally by the emitted light profile. Complete ionization of the deuterium plasma may be obtained using r.f. induction heating alone, with the highest electron temperatures occurring for magnetic fields corresponding to ion cyclotron wave resonance. Operation of the B–65 machine as a magnetic mirror confinement device with ion cyclotron heating led to neutron production, induction coil loading, emission of deuterium and impurity light similar to stellarator operation.

I SHOULD like to present a brief summary of experiments which were made over the past year on the ion cyclotron heating of a plasma (STIX and PALLADINO, 1958; STIX, 1958a). The experiments were done by PALLADINO and myself at Project Matterhorn in Princeton, using the B–65 Stellarator to provide magnetic confinement of the hot plasma. Before going into detail on the experiments, it will be worthwhile to go quickly over the principles and theoretical results which have been obtained up to now (ALFVÉN, 1942; ÅSTRÖM, 1951; STIX, 1957; STIX, 1958b; STIX, to be published).

The geometry which we may consider initially is a cylinder of plasma of uniform density immersed in a uniform axial magnetic field. An induction coil is wrapped around the plasma cylinder, and the induction coil is energized with current which oscillates at the ion cyclotron frequency. An azimuthal electric field oscillating at the ion cyclotron frequency is induced in the plasma. If the plasma density is sufficiently low, the collective effects are negligible and the electric field in the plasma is essentially the vacuum electric field of the induction coil. The ions are accelerated by the electric field, and their orbits are growing spirals. The ions gain energy continually and the plasma is heated.

As the density is increased, collective effects become more important. The electrons are tied to the lines of force and are essentially stationary, but if we consider the radial component alone of the ion motion, we note that we have an ion space-charge cloud which is moving radially in and out. A radial electric field is thereby set up, which perturbs the ion and electron motions in such a manner that the ions are no longer accelerated by the field of the induction coil and the plasma is no longer heated.

One may, however, reduce the ion space-charge field by placing a second induction coil close to the first one, and exciting the second induction coil 180° out of phase with the first. Now when the ions move radially outward underneath the first coil, they move radially inward underneath the second coil, and *electrons* may move back and forth along the magnetic lines of force between the two coils, and neutralize the two ion space charges.

A mathematical analysis based on the linearized Boltzmann-Vlasov equation shows that the plasma now has only an extremely weak resonance at the single particle ion cyclotron frequency, but that there is a strong resonance at the same frequency if the magnetic field is somewhat increased. This resonance corresponds to a type of plasma wave which we call an ion cyclotron wave. This ion cyclotron wave is actually the short-wavelength version of the extraordinary ALFVÉN hydromagnetic wave.

We may use our induction coil to excite ion cyclotron waves in the plasma. Calculations show that transfer of power between the induction coil and the plasma wave should be very efficient. The next problem is to transform the kinetic energy of the ion cyclotron wave into heat energy. There are several ways to cause this transformation. The ordinary process of ohmic dissipation, which takes place via ion-electron collisions, will damp the waves and transform the wave energy into heat. The damping is, however, very slow. A faster transformation is obtained

VOL
2
196

through collisions between ions which have a different charge to mass ratio. One may, for instance, bring about damping in a deuterium ion cyclotron wave by introducing a small amount of hydrogen. The most effective damping, however, is a collisionless process which is akin to Landau damping, and which is called cyclotron damping. An ion cyclotron wave can be forced to undergo cyclotron damping by causing the wave to propagate axially into a region of slightly lower magnetic field strength, or, under some conditions, by causing the wave to propagate across lines of force into a region of higher plasma density. This second condition, in fact, will occur naturally in an experimental device as the induction coil excites an ion cyclotron wave on the outside of a plasma, and the wave travels into the plasma into a higher density region. Absorption of the wave should take place in a layer of plasma where the density reaches the value

$$n_i = 2 \times 10^{16} \frac{A}{\lambda^2 Z^2} \left(1 - \frac{B_0^2}{B^2}\right),$$

where n_i is the ion density in cm^{-3}, A and Z are the atomic number and charge, λ is the axial wavelength of the ion cyclotron wave in cm, B is the magnetic field strength and B_0 is the magnetic field strength for which the ion cyclotron frequency is equal to the generator frequency.

Experiments on ion cyclotron heating have been done on the B–65 Stellarator, shown in Fig. 1. The gas pressure was 1 micron of deuterium, and the r.f. generator frequency was 11·5 megacycles, which is equal to the deuterium ion cyclotron frequency at 15·1 kilogauss. The r.f. generator puts about 200 kilowatts of power into the plasma. Our principal diagnostic techniques to date have been spectroscopy and neutron detection. For the optical spectroscopy, we are able to view the discharge axially, through the 'longview' port, and also transversely, looking directly into the quartz tube inside the induction coil in the ion cyclotron heating section. Furthermore, by means of a movable mirror in the light path for the transverse viewing, we are able to scan across the plasma, and obtain data for the radial distribution of the light emission.

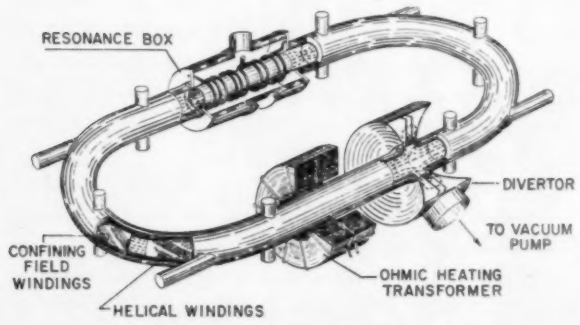


FIG. 1.—Ion cyclotron heating experiment.

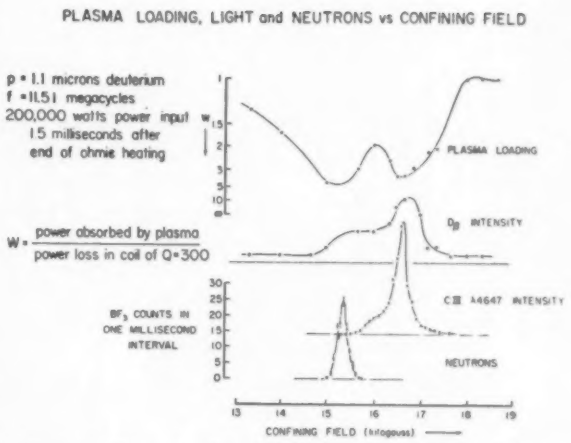

PLASMA LOADING, LIGHT and NEUTRONS vs CONFINING FIELD
p = 1.1 microns deuterium
f = 11.51 megacycles
200,000 watts power input
1.5 milliseconds after end of ohmic heating
W
PLASMA LOADING
W = power absorbed by plasma / power loss in coil of Q=300
D_β INTENSITY
CIII 4647 INTENSITY
BF_3 COUNTS IN ONE MILLISECOND INTERVAL
NEUTRONS
CONFINING FIELD (kilogauss)

FIG. 2.—Plasma loading, light and neutrons vs. confining field.

Fig. 2 shows probably the most important experimental results achieved so far. Several measured quantities are plotted against the value of the confining magnetic field. The top trace shows the measured electrical loading on the induction coil, and it is seen that two peaks appear in the loading. At the first peak, neutrons appear. The value of the confining field is, within the error of measurement, equal to the single particle cyclotron resonance field, B_0. The simplest interpretation is that the neutrons come from deuterons which have been accelerated in the low density region at the outer edge of the plasma, where the electric field is strong and is approximately equal to the vacuum field of the induction coil, as discussed earlier. At a larger value of confining field, deuterium and carbon light appear. The emission of these spectra is due to a more general heating of the plasma than takes place at the single particle resonance. The presence of this heating is evidence first for the ion cyclotron wave phenomenon, which is predicted to occur for values of magnetic field larger than the single particle resonant field. Second, the heating is evidence for the absorption of ion cyclotron wave energy, and its transformation into heat.

More recent experiments have been directed toward the detailed examination of the ion cyclotron wave absorption. As mentioned earlier, the wave is expected to propagate radially inward and to be absorbed when it reaches a region of critical density. For B–65 at the higher loading peak, the critical density is about 10^{13} cm^{-3}, whereas the density in the centre of the tube is probably three to five times larger. The

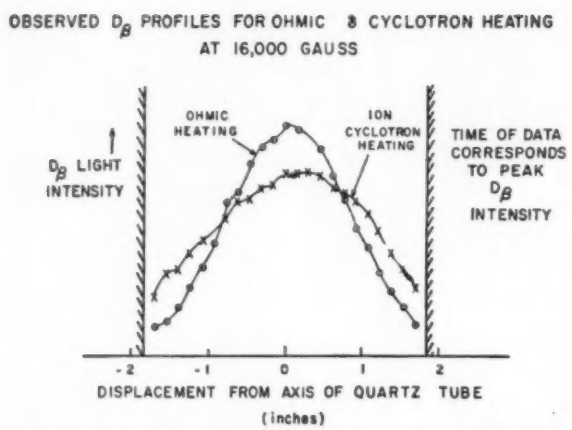


FIG. 3.—Observed $D\beta$ profiles for ohmic and cyclotron heating at 16,000 gauss.

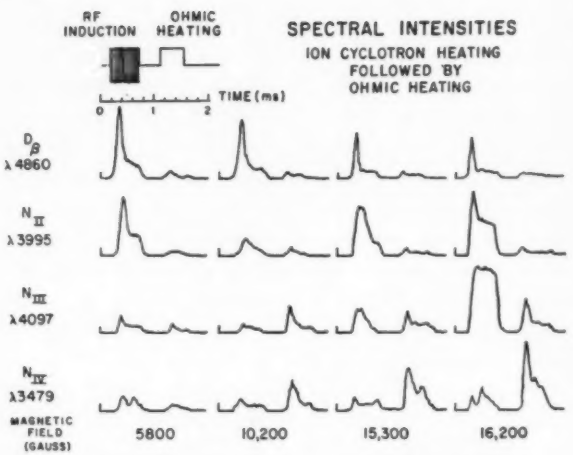


FIG. 4.—Spectral intensities. Ion cyclotron heating followed by ohmic heating.

energy absorption should take place at some intermediate value of radius. In Fig. 3, we see a profile of the observed D_β light emission as a function of distance from the tube axis. The data was obtained by a vertical scan of the quartz tube in the induction coil section, looking through the transverse viewing port. For comparison, the ohmic heating profile is also shown. The r.f. profile is seen to be considerably broader than the ohmic heating profile, giving some qualitative support to the critical density theory. However, one would expect the r.f. heating to be strong at the edge of the plasma also because the induced azimuthal electric field is strong there. We have not yet been able to separate the critical density effect from the electric field geometry effect.

The deuterium light shown in Fig. 3 comes from the plasma during the process of ionization of the deuterium at the ion cyclotron wave resonance condition. We have found, in fact, that the r.f. excitation is sufficient in itself to ionize the deuterium plasma completely. Spectroscopic evidence for this from the transverse viewing port is presented in Fig. 4. A 500 μsec pulse of r.f. induction heating is followed by 300 μsec of ohmic heating. Any deuterium remaining unionized in the stellarator after the r.f. pulse is ionized by the ohmic heating current, with the consequent emission of D_β spectral light. The oscilloscope traces show that not only does a negligible amount of deuterium light come from the ohmic heating pulse when the magnetic field corresponded to ion cyclotron wave resonance (16 kilogauss), but also when the magnetic field took on lower values. Impurity light shows up, however, during the ohmic heating pulse. An examination of the light from nitrogen in different stages of ionization shows that the highest electron temperatures are reached with the r.f. induction heating when the ion cyclotron wave resonance condition obtains. Somewhat higher electron temperatures are reached with ordinary ohmic heating.

The ion temperature of the plasma with ion cyclotron wave resonance heating may be inferred from measurements of the Doppler broadening of the impurity lines. From the C III λ 4647 line, an ion temperature of 50 to 80 volts was obtained looking into the 'longview' port through the resonance section, while a temperature twice that high was obtained from the transverse viewing into the resonance section. This difference in temperature is consistent with the simple view that the resonance section is the source of heat, and that the heat diffuses out into the rest of the stellarator from there.

Looking back at Fig. 3, we note again that the ion cyclotron heating light came from a wider source than the ohmic heating light. However, the ohmic heating light may be taken as a measure of the true magnetic aperture of the stellarator. The use of helical windings makes this aperture considerably smaller than the tube diameter. We concluded that much of the r.f. heating took place outside of the aperture of the stellarator. In a confirming test, the stellarator aperture was destroyed entirely by failing to energize the confining field in the two U-bends. Magnetic mirrors on each end of the resonance section still provided plasma confinement, and neutron production, deuterium and impurity light emission and induction coil loading were obtained with characteristics almost identical to stellarator operation.

We have just begun experiments with the B–66 stellarator at Princeton. This machine is slightly larger than B–65, but has a baked ultra-high vacuum system (B–65 has O-ring seals) and incorporates a 'magnetic beach' geometry which is designed to produce thermalization of ion cyclotron waves propagated axially into

VOL 2 196

a decreasing magnetic field. The magnetic beach makes possible in principle the absorption of ion cyclotron wave energy at higher densities than may be reached with 'critical density' absorption.

REFERENCES

ALFVÉN H. (1942) *Ark. Mat., Astr. Fys.* **29B,** No. 2.

ÅSTRÖM E. (1951) *Ark. Fys.* **2,** 443.

STIX T. H. (1957) *Phys. Rev.* **106,** 1146.

STIX T. H. (1958a) *Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva,* **31,** 282. United Nations, N. Y.

STIX T. H. (1958b) *Phys. Fluids* **1,** 308.

STIX T. H. *Absorption of Plasma Waves.* To be published.

STIX T. H. and PALLADINO R. W. (1958) *Phys. Fluids* **1,** 446.

VOL.
2
961

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 88 to 89. Pergamon Press Ltd. Printed in Northern Ireland

TRAVELLING WAVE INTERACTION IN PLASMAS

G. D. BOYD* and R. W. GOULD
California Institute of Technology, Pasadena, California

Abstract—Interaction between a slow space charge wave which travels along a cylindrical plasma column and an electron beam which passes down the axis of the column is demonstrated experimentally. A spatially growing wave exists when the velocity of the beam is approximately equal to the velocity of the unperturbed wave.

IT has been shown by a number of investigators that a plasma column of finite radius supports a space charge or 'electrostatic' wave which travels along the column with a phase velocity which, under suitable conditions, can be made to be only a few hundredths of the velocity of light.† This wave propagates at frequencies well below the electron plasma frequency, ω_p, and has an upper cut-off at $\omega = \omega_p/\sqrt{1+K}$, where K is the dielectric constant of the material (glass or vacuum) immediately surrounding the plasma column. Because this wave has a strong longitudinal component of electric field, an electron beam travelling down the axis of the column should interact strongly with the wave when the electrons travel at approximately the same speed as the wave (GOULD and TRIVELPIECE, 1958). Electrons are bunched by the field of the wave and the bunches in turn reinforce the wave. The amplitude of the wave grows as it progresses down the plasma column. This is precisely the travelling wave tube interaction except that the radio frequency fields acting on the electron beam are those of the plasma wave instead of those of a wave travelling along a helical conductor. The theory of travelling wave interaction (PIERCE, 1950) may therefore be applied to calculate the rate of spatial growth.

This paper reports the results of an experiment designed to verify this interaction in a cylindrical plasma column. A schematic drawing of the tube used is shown in Fig. 1. The plasma is generated by an arc discharge in mercury vapour and the positive column of the discharge is used for the plasma interaction region. The arc cathode on the right is the source of plasma electrons. The electron gun on the left produces a beam which is collected by the beam collector on the right. Modulation at a frequency of 490 Mc is introduced on the electron beam by means of the input cavity resonator on the left. Currents in the beam and in the plasma induce the signal in the output cavity resonator on the right. In addition, a travelling radio frequency probe (not shown in the figure) which could be moved along the interaction region, was used to determine how the wave amplitude varied along the column.

Fig. 2 shows the nature of the results which were obtained. The oscilloscope traces (above) show the output signal as a function of current in the arc discharge. In a low pressure discharge the electron density is proportional to discharge current, hence the abscissa of these traces is proportional to the square of the electron plasma frequency in the interaction region. The oscilloscope traces show that for each beam voltage there is a plasma density which produces a greatly increased output signal. At this point the plasma electron density is such that the velocity of the wave which travels along the column is approximately equal to the beam velocity. It may be seen from the sequence of traces that as the beam velocity is increased by increasing the accelerating potential, the density required for velocity synchronism increases. From this and similar data the dispersion characteristic of the plasma wave can be determined in the manner described below.

Traces in the lower half of Fig. 2 show the variation

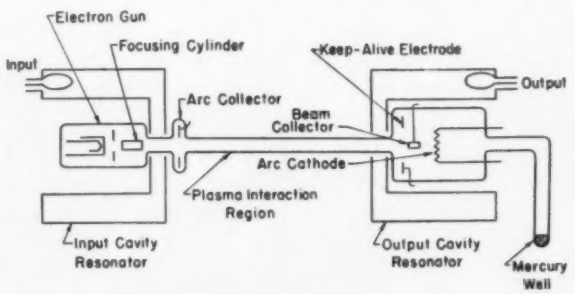


FIG. 1.—Schematic drawing of travelling wave interaction experiment.

* Present address: Bell Telephone Laboratories, Inc., Murray Hill, New Jersey.

† A detailed discussion of this wave as well as a list of references to prior papers is contained in TRIVELPIECE and GOULD (1959). (See also LINHART J. *Proceedings of the California Institute of Technology.*

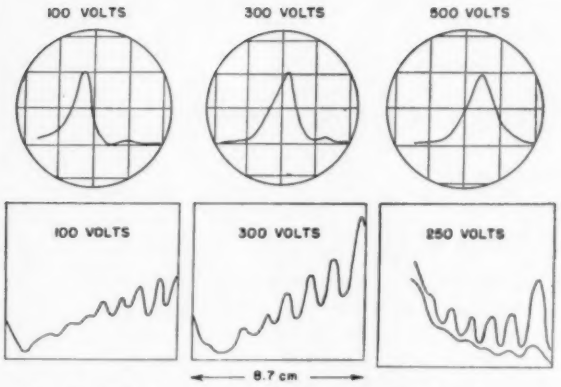


FIG. 2.—Signal output vs. discharge current for electron beam voltages: 100V, 300V, and 500V, (oscilloscope traces). Signal level along plasma column vs. distance from input for electron beam voltages: 100V, 300V, 250V, (probe traces).

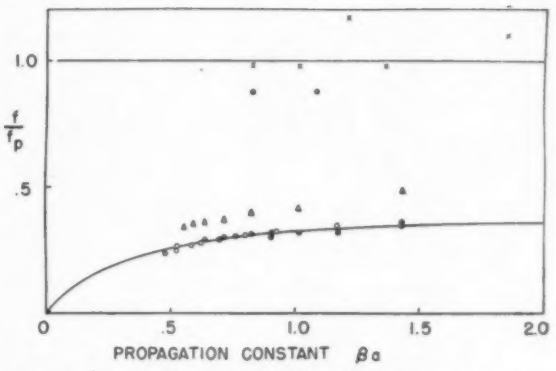


FIG. 3.—Propagation characteristics of plasma column as determined by electron beam interaction (experimental points). Solid curves indicate theoretical results (TRIVELPIECE and GOULD, 1959).

in signal amplitude along the tube, as measured by the travelling radio frequency probe. In each trace the discharge current has been adjusted so as to produce the maximum signal in the output cavity resonator. In the first two traces *growth* of the signal is apparent as the wave propagates to the right. The periodic variations in the amplitude are due to the interference with the wave which is reflected from the output cavity resonator. The spatial periodicity of the variations is a half wavelength of the plasma wave and clearly increases as the velocity of the wave is increased (corresponding to increased beam potential in these traces). The third trace is a double one, the lower of the two traces being obtained with the electron beam off. The attenuation of the wave which is observed there is due to collisions of the plasma electrons with ions, neutral molecules and the wall of the discharge tube. Introduction of the electron beam (upper trace) reduces the attenuation but the interaction is not sufficiently strong to produce much growth.

The data of Fig. 2 and other similar data has been used to construct the dispersion characteristic of the space charge wave of the cylindrical plasma column. Assuming that the maximum output corresponds to velocity synchronism between the wave and the beam one then knows both the frequency of the wave ($f = \omega/2\pi$) and the velocity (calculated from the accelerating potential of the electron beam, $v = \sqrt{2eV/m}$.) The axial wave number is then obtained from the relation $\beta = \omega/v$. The result of a number of measurements with different accelerating voltages is shown in Fig. 3 along with the theoretical result of TRIVELPIECE and GOULD (1959), (solid line). The electron plasma frequency $f_p = \omega_p/2\pi$, was measured independently by the cavity perturbation technique (BIONDI and BROWN, 1949). The two rather distinct sets of data were obtained with different adjustments of the various electrode potentials. The difference between the two data reflects the fact that the constant of proportionality between electron density and discharge current depends slightly upon electrode potentials, whereas the *same* constant was actually employed in calculating f/f_p for the two runs.

Also shown in Fig. 3 is the interaction at plasma resonance ($f = f_p$) of the type which has previously been reported (1958). As in the travelling wave interaction at lower values of f/f_p, a pronounced increase in the signal level at the output cavity resonator is observed. Although the radio frequency fields outside the plasma are very weak, it has also been possible to confirm that this increase in output corresponds to a spatially growing wave. The interaction between an electron beam and a plasma medium at plasma resonance is essentially the one described by BOHM and GROSS (1949a, 1949b and 1950) and others, though an extension of the one-dimensional theory is necessary to account for the finite beam diameter.

Acknowledgements—It is a pleasure to acknowledge valuable discussions with L. M. FIELD, A. W. TRIVELPIECE and D. G. Dow and the support of the United States Navy Office of Naval Research in conducting this investigation.

REFERENCES

BIONDI M. A. and BROWN S. C. (1949) *Phys. Rev.* **75,** 1700.
BOHM D. and GROSS E. P. (1949a) *Phys. Rev.* **75,** 1851; (1949b) **75,** 1864; (1950) **79,** 992.
BOYD G. D., FIELD L. M. and GOULD R. W. (1958) *Phy. Rev. (Letters)* **109,** 1393.
GOULD R. W. and TRIVELPIECE A. W. (1958) *Proc. Polytech. Inst.* Brooklyn Symposium.
PIERCE J. R. (1950) *Travelling-Wave Tubes,* Van Nostrand, New York.
TRIVELPIECE A. W. and GOULD R. W. (1959) *J. Appl. Phys.* **30.**

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 90 to 93. Pergamon Press Ltd. Printed in Northern Ireland

CYCLOTRON RADIATION FROM A HOT PLASMA*†

W. E. DRUMMOND

John Jay Hopkins Laboratory for Pure and Applied Science, General Atomic Division of General Dynamics Corporation, San Diego, California

Abstract—In their Geneva paper, TRUBNIKOV and KUDRYAVTSEV calculated the cyclotron radiation from a hot plasma. In doing this, the approximation was made that the individual particles radiated as though they were in a vacuum. We have investigated this approximation by calculating the absorption length directly from the Boltzmann equation and we find that indeed this approximation is correct whenever $\omega_p^2/\Omega^2 \ll n^2$ where n is the harmonic number of the radiation in question, ω_p is the plasma frequency, and Ω is the cyclotron frequency. For a contained plasma, the left-hand side of this inequality is of the order of magnitude of one and thus the inequality is well satisfied for the dominant radiation from a plasma at high temperature.

The physical reason for this inequality can be investigated by solving the test-charge problem to find the transverse response current of the plasma to the motion of an electron. The result is that the transverse organizing length in the plasma is c/ω_p, and thus one may picture an electron in its Larmor orbit as being surrounded by a co-moving current cloud of radius c/ω_p. Classical electrodynamics then leads to the above inequality.

AT the *Geneva Conference on the Peaceful Uses of Atomic Energy*, a paper by TRUBNIKOV and KUDRYAVTSEV (1958) on the cyclotron radiation from a hot plasma was given. Their results were that the cyclotron radiation is unexpectedly large and that it is essentially a surface effect and can be minimized with respect to volume effects such as thermonuclear-energy production by simply increasing the size of a controlled thermonuclear reactor. Their estimates of the critical size of a controlled thermonuclear reactor, i.e. the size at which the thermonuclear energy production is just balanced by the radiation energy loss, were in the neighbourhood of 10–100 meters.

Because of the serious ramifications these results have on the future of controlled thermonuclear power considerable controversy has arisen over their method of calculation (see e.g. BEARD, 1959). In particular they assume that an individual electron in a plasma will radiate as though it is in a vacuum and this assumption has been challenged by several people. For example, it has been suggested that each electron in a plasma is surrounded by a 'Debye cloud' of charge which would inhibit the radiation from this electron. Rosenbluth and I have considered this problem and conclude that indeed an electron is surrounded by a co-moving cloud of current, but the important part of this current cloud for inhibiting radiation is the transverse part and we find that the transverse part has a radius of approximately c/ω_p. This will inhibit radiation at wavelengths long compared to c/ω_p but radiation at wavelengths short compared to c/ω_p will be essentially unaffected. This means that for frequencies large compared to the plasma frequency an electron in a plasma does indeed radiate as though it were in a vacuum and the Russian assumption is correct if the frequencies of interest, i.e. multiples of the cyclotron frequency, are large compared to the plasma frequency. To show that this is so we consider the quantity $\beta_H = 8\pi nkT/B^2$. For plasmas of interest β_H is a number between 1/10 and 1 and this means that for temperatures in the neighbourhood of 50 to 200 kilovolts the plasma frequency and the cyclotron frequency are roughly equal. It follows that multiples of the cyclotron frequency are large compared to the plasma frequency. Thus we agree with the basic assumption of TRUBNIKOV and KUDRYAVTSEV.

Before proceeding to a derivation of our results, I wish first however to give a short summary of the method used by TRUBNIKOV and KUDRYAVTSEV. Consider an electron moving in a circular orbit the radius of which is given by the Larmor formula $r = |v_\perp/\omega_c|$ where $v_\perp$ is the magnitude of the projection of the velocity vector on to the plane normal to **B**. To calculate the vector potential of such an electron we have the expression

$$A(\omega) \propto \int \exp(i\omega t) \exp[i\mathbf{k} \cdot \mathbf{r}(t)] v(t) dt.$$

In many radiation problems one makes the dipole approximation, i.e. one neglects $|kr|$ compared to 1. For this case we see that we obtain only radiation at

* This work was done under a contract between General Atomic and the Texas Atomic Energy Research Foundation.

† The work reported in this talk was done in co-operation with M. ROSENBLUTH.

VOL 2 196

the cyclotron frequency. For our case however $|kr| = |(\omega/\omega_c)(v_\perp/c)|$ which cannot be neglected compared to 1 for a hot plasma. Thus it is the failure of the dipole approximation that is responsible for the harmonics of the cyclotron frequency. What TRUBNIKOV and KUDRYAVTSEV did was to calculate from this the energy radiated from a single electron and to average this over a Maxwellian distribution of electrons to obtain the source function S which is the energy radiated per second per unit solid angle per unit frequency per unit volume. Now the cyclotron frequency of a relativistic electron depends upon the electron momentum. Thus, when we average over a Maxwellian distribution of electrons, we smear out the spectrum of radiation into a continuum. From Kirchhoff's law we obtain a relation between the source function S and the absorption coefficient α, $S = I_{RJ}\alpha$ where I_{RJ} is the Rayleigh-Jeans distribution law. Thus one obtains in a simple way the absorption coefficient of the plasma. For low frequencies where there is considerable radiation there will also be considerable absorption and a plasma of finite thickness will appear 'optically thick.' As the frequency is increased to very high harmonics of the cyclotron frequency the source function and absorption coefficient will gradually drop off until the plasma is optically thin. Thus a plasma of finite thickness appears to be a black body for all frequencies up to the limiting frequency ω^* at which the absorption length is roughly equal to the plasma thickness. This is the result of TRUBNIKOV and KUDRYAVTSEV.

We now wish to discuss the philosophy behind our derivation. If one considers the limit in which $e = m = 1/n = T = 0$ with e/m, etc. constant, the collisionless Boltzmann equation is an exact result of the Liouville theorem. This is the fluid limit and there are no individual-particle effects such as cyclotron radiation in this limit. If we consider e, m, $1/n$ and T all to be of the same order (we refer to the order of any of these quantities as g) then cyclotron radiation appears as a first order effect in g. The straightforward way to calculate this is to introduce a test charge into the plasma. Besides the current of the test charge there will also be the reaction current of plasma. This includes effects due to two-particle correlations as well as the fluid response. This total current is then the source of cyclotron radiation. The total radiation is then given by averaging over a Maxwellian distribution of test charges and the source function so derived is of order $ne^2 = g$. A simpler way of obtaining the results is to calculate the absorption coefficient to zeroth order in g, i.e. from the collisionless Boltzmann equation, and then obtain the source function from Kirchhoff's law which, because I_{RJ} is of first order in g relates quantitites of different orders. This is what we have done in the following.

Thus we wish to find the absorption coefficient for electromagnetic waves from the collisionless Boltzmann equation. Because we are dealing with very hot plasmas we use the relativistic Boltzmann equation. If we linearize, Fourier transform with respect to space, and Laplace transform with respect to time, we obtain

$$\left[s + \frac{i\mathbf{k}\,.\,\mathbf{p}}{\sqrt{1+p^2}} + \frac{e}{\sqrt{1+p^2}}(\mathbf{p} \times \mathbf{B})\,.\,\nabla_p\right] f = -e\mathbf{E}\,.\,\nabla_p f_0 + b.$$

Here s is the Laplace transform variable and k is the Fourier transform variable. The solution to this equation can be written formally as

$$f(k,\omega,\mathbf{v}) = \mathbf{G}\,.\,\mathbf{E}(k,\omega)$$

where $\mathbf{G}$ is an operator and $m = c = 1$. Similarly from Maxwell's equations we obtain

$$(s^2 + k^2)\mathbf{E} - (k\,.\,E)\mathbf{k} + 4\pi s\mathbf{j} = \mathbf{b}$$

where $\mathbf{j}$ the current is defined by

$$\mathbf{j} = \int \mathbf{v} f(\mathbf{v}) d^3v = \int \mathbf{v}\mathbf{G}\,.\,\mathbf{E} d^3\mathbf{v} = \boldsymbol{\sigma}\,.\,\mathbf{E}$$

and b, $\mathbf{b}$ are concerned with the initial data and are of no interest for the present problem. Thus we obtain an equation of the form

$$\mathbf{A}\mathbf{E} = \mathbf{b}$$

where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$

$$= \begin{pmatrix} s_2 + 4\pi s\sigma_{11} & 4\pi s\sigma_{12} & 0 \\ 4\pi s\sigma_{21} & s^2 + c^2k^2 + 4\pi s\sigma_{22} & 0 \\ 0 & 0 & s^2 + c^2k^2 + 4\pi s\sigma_{33} \end{pmatrix}$$

The dispersion relations are obtained by setting the determinant of $\mathbf{A}$ equals to zero. For propagation directly across the magnetic field the 13, 23, 31, 32 elements of $\mathbf{A}$ are zero because of symmetry. We are interested in propagation at very small angles to the normal of the magnetic field and thus we neglect these elements. The dispersion relations are then obtained from the equation $a_{33}(a_{11}a_{22} - a_{12}a_{21}) = 0$. For propagation across the magnetic field with polarization along the field, the dispersion relation is given by $a_{33} = 0$. For propagation across the magnetic field with polarization perpendicular to the magnetic field, the situation is complicated by the off-diagonal

elements of the conductivity tensor. However it is easy to show that these elements can be neglected compared to the diagonal elements for frequencies large compared with the plasma frequency and as we have seen, these are the frequencies of interest. For such waves the dispersion relationship is then simply given by

$$a_{22} = 0.$$

Actually it is this polarization which is the most important. The imaginary part of σ_{22} leads simply to a slight frequency shift but the real part gives us damping in time and this is related to spatial absorption by the group velocity. In turn, this is related to the source function through Kirchhoff's law and thus to find the source function we must simply evaluate the real part of σ_{22}. An elementary calculation gives us

$$S(\omega,\mathbf{\Omega}) = I_{RJ}\,\alpha(\omega,\mathbf{\Omega}) = \int f_0(p)d^3p \times\left[\frac{e^2\omega^2}{2\pi} r_\perp{}^2 \sum_m J_m{}'^2(k_\perp r_\perp)\right]$$

where the quantity in brackets is simply the energy radiated by a single particle in a vacuum, and $k_\perp$ is the magnitude of the projection of the propagation vector onto the plane normal to B and r is $v_\perp\omega_c$. And thus we see, by direct calculation, that the source function is indeed just the source function for independent electrons averaged over a Maxwellian distribution. TRUBNIKOV and KUDRYAVTSEV have evaluated this integral for the case of propagation exactly across the magnetic field. We have extended this to the case of small angles from the normal magnetic field and obtain

$$\alpha_\perp = \frac{\omega_p{}^2}{4\Omega}\sqrt{\frac{\pi}{T}}(\epsilon^2-1)^{3/2}\left(\frac{1}{mT}\right)^2 \times \exp\left\{\frac{1}{T} - \frac{2m}{\epsilon^2-1} - m\cos^2\theta\right\}$$

where

$$m = \frac{\omega}{\omega_c}$$

and

$$\frac{2\epsilon}{\epsilon^2-1} - \ln\frac{\epsilon+1}{\epsilon-1} = \frac{1}{mT}$$

which agrees with results of TRUBNIKOV and KUDRYAVTSEV for $\theta = \pi/2$. It is important to recognize here that the radiation is almost entirely contained within an angular region of the order of $1/\sqrt{m}$ about the normal to the magnetic field. And this means that the black-body spectrum will not fill up at large angles to the normal. If one now calculates the radiation from a slab of plasma of thickness L a considerable reduction occurs compared to the results of TRUBNIKOV and KUDRYAVTSEV because of this angular effect and this materially changes the critical size. Equating the total radiation out to the thermonuclear energy production we then obtain the table of critical sizes shown in Table 1.

Note added in galley.* *Due to an algebraic mistake in the evaluation of $\alpha_\perp$, the angular distribution given above is in error. Actually $\alpha_\perp$ falls off less strongly with angle and the results given in Table* 1 *are wrong and it has thus been deleted. Although it has not yet been evaluated quantitatively there should nevertheless be some reduction in the critical size, but not as much as previously stated.*

It results that for the case of $\beta_H = 1$ cyclotron radiation is not too bad a problem. However, for stability reasons it may be desirable to have β_H in the neighbourhood of 1/10 and for this case the critical size again becomes large. In order to mitigate the effects of cyclotron radiation one can use reflectors. If one considers a reflector with reflectivity R which does not change the angle the radiation makes with the magnetic field, the effect of the reflector can be analysed as follows (see Fig. 1):

$$\frac{dI}{dz} = -\alpha I + S = -\alpha(I - I_{RJ}).$$

The boundary conditions are

$$I(1) = I(3) = RI(2)$$

and these then yield

$$(1-R)\,I(L) = (I - R)\,I_{RJ}\,\frac{(1-e^{-\alpha L})}{(1-R^{-\alpha L})}.$$

Inserting this into the critical equation one then finds that the critical size is reduced by the factor 1–R. For magnetic fields in the neighbourhood of 10^4 gauss the harmonics of the cyclotron radiation are in the millimetre and centimetre range and a reflectivity of 0·99 is not unreasonable for such a case. Thus by using a reflector the critical size can be reduced by two orders of magnitude.

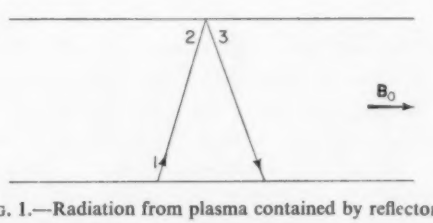


FIG. 1.—Radiation from plasma contained by reflector.

* This was pointed out to us independently by TRUBNIKOV and by HUBBARD.

To sum up I might say that we have shown three things. First, that the basic assumption of TRUBNIKOV and KUDRYAVTSEV, i.e. that electrons in a plasma radiate as though they are in a vacuum, is correct. Second, that the angular dependence of the cyclotron radiation leads to some reduction in the critical size of a controlled thermonuclear reaction compared to the results of TRUBNIKOV and KUDRYAVTSEV. Third that machines such as the mirror machine which operate at low β will necessarily use reflectors and that such reflectors can be expected to cut down the critical size by two orders of magnitude or more.

REFERENCES

BEARD D. B. (1959) *Phys. Fluids* **2**, 379; **3**, 324.

TRUBNIKOV B. A. and KUDRYAVTSEV V. S. (1958) *Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva*, **31**, 93. United Nations N.Y.

VOL.
2
961

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 94 to 97. Pergamon Press Ltd. Printed in Northern Ireland

OPTICAL PROPERTIES AND EMISSION OF RELATIVISTIC PLASMAS AT CYCLOTRON RESONANCE

D. B. BEARD
University of California, Davis, California

Abstract—The cyclotron emission from a hot, completely ionized, magnetically-confined plasma has been estimated by computing the absorption of an incident plane wave. The harmonics of the fundamental cyclotron frequency, which are emitted perpendicular to the magnetic field direction, were summed over and also treated individually. Because of the Doppler effect and the relativistic variation in mass the behaviour of the electrons and the electro-magnetic properties of the medium are functions of electron velocity. Assuming a Maxwell–Boltzmann distribution in electron velocity, the polarization of the plasma was computed by integrating over electron velocity. Solution of Maxwell's equations yielded the optical constants of the plasma and thereby the absorption of an incident wave. By invoking Kirchhoff's relation, the emission of the plasma was then determined.

SOME very fine work has been done on this subject by BERNSTEIN (1958), DRUMMOND (1958a and 1958b) and others using transform techniques on the Boltzmann equation as LANDAU has done. TRUBNIKOV and KUDRYAVTSEV (1958) have also treated the cyclotron emission by adding up the total emission perpendicular to the magnetic field from all electrons and comparing this total emission to a black body. The emission of cyclotron radiation may also be treated by examining the behaviour of electromagnetic radiation incident on the plasma in a way differing from that DRUMMOND has just discussed so well. Among its several disadvantages this method also has the advantage of directly assessing the surface emissivity, that is, of directly determining how much of the radiation gets out and how much is contained by a mirror-like plasma surface layer. A second advantage lies in the physical insight gained from examining the problem in what might be considered as a more direct way.

The problem is treated by examining what happens to an incident electromagnetic plane wave of given frequency, ω, after it enters a plasma of electron temperature, T, contained in a magnetic field, H. From the absorption of this plane wave, the plasma emission of the frequency, ω, may be readily computed by comparison to a black body at the same temperature.

Following LORENTZ (1952) we write the equation of motion of an electron confined by a magnetic field in the presence of an incoming time dependent electric wave $\mathbf{E}_0 \exp i\omega t$

$$\frac{d}{dt}\mathbf{P} = e\mathbf{E} - g\mathbf{v} + \frac{e}{c}\mathbf{v} \times \mathbf{H}. \qquad (1)$$

g is a velocity-dependent damping term $= 2mv_c$ where v_c is the collision frequency. It need not necessarily represent energy loss; it also accounts for the line broadening resulting from collisional interrupting of a wave train of given phase. If one were to let the polarization vector $\mathbf{P} = Ne\mathbf{r}$ and substitute $\mathbf{P}$ for $\mathbf{r}$ in equation (1), one would then find a resonance in the polarization vector occurring at $\omega_0 = eH/mc$ from the last term (BEARD, 1959). That is, the last term merely causes the medium to respond as though it consisted of oscillators of natural frequency $\omega_0 = eH/mc$.

If this equation is to hold in the laboratory frame, relativistic effects must be noted if the electron is moving relativistically in the laboratory frame. Consequently

$$\frac{d\mathbf{P}}{dt} = \frac{m}{\sqrt{1 - v^2/c^2}}\frac{d\mathbf{v}}{dt} + \frac{m\mathbf{v}}{c^2(1 - v^2/c^2)^{3/2}}\frac{d\mathbf{v}}{dt} \qquad (2)$$

Because of the electron motion in the laboratory frame there is a Doppler shift in frequency of the incident radiation as seen by the electron. $\omega^1 = (1 - v^2/c^2)^{-1/2}(1 + \mathbf{v} \cdot \mathbf{k}/kc)\omega$ where $\mathbf{k}$ is the unit vector in the direction of propagation of the incident wave. The polarization vector $\mathbf{P} = Ne\mathbf{r}$. Pursuing this train of investigation, substituting $\mathbf{P}/Ne$ for $\mathbf{r}$, in equation (1) would require taking account of the circular motion of the electrons. Only electrons having a velocity between $\mathbf{v}$ and $\mathbf{v} + \mathbf{dv}$ would contribute in the same way to the polarization. That is, we might let $\mathbf{P} = N(\mathbf{v})e\mathbf{r}$; where the total

$$\mathbf{P} = \int \{\int P_x \sin\varphi \, d\varphi \mathbf{i} + \int P_y \cos\varphi \, d\varphi \mathbf{j} + \int P_z \, d\varphi \mathbf{k}\} \, dv$$

where φ is the phase of the electron motion. The

components of the total polarization vector would consist of integrals such as

$$\int_0^{2\pi} \exp\,(i\omega t[1 - v^2/c^2]^{-1} \times [1 + (v/c)\sin\theta\sin\varphi])\sin\varphi\, d\varphi.$$

Other modifications occur in the magnetic field term when account is taken of the magnetic field associated with the incident electric wave. This amounts essentially to the Fourier components of the incident radiation with the electron frequency $\omega_0 = eH/mc$ as the fundamental Fourier frequency.

Alternatively the problem may be simplified by noting that LANDAU and SCHWINGER have already solved much of this part of the problem by computing the Fourier analysed emission spectrum of a single electron. One need then only treat the electron gas as a collection of polarized oscillators interacting with radiation according to the formulae given by LANDAU and LIFSHITZ (1951) and SCHWINGER (1949). The oscillators resonate not at just one frequency but at multiples of a fundamental frequency $\omega_0' = \omega_0\gamma = \frac{eH}{mc}\sqrt{1 - v^2/c^2}$. The Z-axis is taken to be parallel to the confining magnetic field throughout this paper. For propagation perpendicular to the magnetic field, corresponding to equation (1) for each oscillator one now has instead of the $e\mathbf{E}$ term for an oscillator of resonant frequency $l\omega_0'$

$$e(S_{lx}E_x\mathbf{i} + S_{ly}E_y\mathbf{j} + E_z\mathbf{k}) \qquad (3)$$

where

$$S_{lx} = (c/v)\,J_l(lv/c), \quad S_{ly} = J_l'(lv/c).$$

Note that for $v^2 \ll c^2$

$$S_{lx} \sim S_{ly} \sim \frac{1}{2(l-1)!}(lv/2c)^{l-1} \sim (\exp/2) \times (2\pi[l-1])^{-1/2}(\exp v/2c)^{l-1} \qquad (4)$$

If the contribution to the ith component of the polarization made by an oscillator with velocity v and frequency $l\omega_0'$ is represented by P_{lx} then $\mathbf{r} = \mathbf{P}_l/eN(v)$ and $\mathbf{P} = \int_0^\infty \sum_{l=1}^\infty \mathbf{P}_l\, dv$,

$$-m\omega^2\mathbf{P}_l/Ne = e(S_{lx}E_x\mathbf{i} + S_{ly}E_y\mathbf{j} + E_z\mathbf{k}) - i\omega g\mathbf{P}_l/Ne + im\omega l\omega_0\mathbf{P}_l \times \mathbf{H}/NeH. \qquad (5)$$

Except for a Doppler shift in the Z-direction the equation for propagation parallel to the magnetic field is equation (1). Two extreme cases will be treated.

Case 1. *Propagation along the magnetic field lines.* $\sin\theta = 0$, $S_{li} = \delta_{l,1}$. The solution to equation (5) is in this case $P_y = \pm iP_x$, $P_z = 0$, and

$$P_x = eE_x\left\{\frac{-m\omega^2}{\sqrt{1 - v^2/c^2}} + \frac{i2mv^e\omega}{\sqrt{1 - v^2/c^2}} \pm m\omega\omega_0(1 - v_z/c)\right\}^{-1} eN(v_p, v_z) \qquad (6)$$

where the $\sqrt{1 - v^2/c^2}$ in the numerator of the ω_0 term cancels the same function in the denominator and the $\left(1 - \frac{v_z}{c}\right)$ factor also results from the Doppler shift of the moving electron. The oscillators are free to move only in the Z-direction.

$$P_z = \frac{e^2E_z}{\omega}\int_{-\infty}^{\infty}\int_0^\infty \frac{N(v_p, v_z)\,dv_p\,dv_z}{-m\omega(1 - v^2/c^2)^{-1/2} + m\omega_0 + m\omega_0(v_z/c) + izmv_c(1 - v^2/c^2)^{-1/2}} = \phi(\omega)E_z \qquad (7)$$

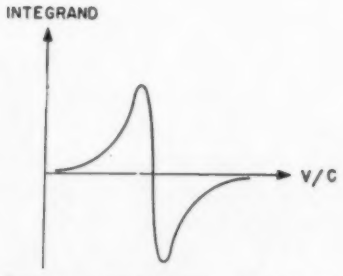


FIG. 1.—Illustration of the behaviour of the real part (the index of refraction contribution) of the polarization integrand as a function of electron velocity.

The real part of the integrand swings from large positive values near resonance to large negative values as v_z or v increases. Typical of refractive index curves the real part of the integrand looks like Fig. 1. The result is that the real part of this integral which will be shown to result in the index of refraction is surprisingly small. The medium is made up of oscillators of continuously distributed resonant frequency which simultaneously advance and retard the phase of the incident wave so that the index of refraction is small although possibly greater than one and changes slowly over a wavelength. Thus ray optics obtains for wave propagation across the medium boundary. That is,

essentially no reflection takes place at the edge of the plasma—only absorption and emission—or transmission if the plasma is optically thin.

I attempted to solve this problem several years ago and in my haste to write this up recently when I learned security restrictions were lifted last December the material was no longer familiar to me and I made several mistakes. I am especially grateful to I. BERNSTEIN and M. ROSENBLUTH for calling my attention to an error I made in evaluating the imaginary part of this integral. If $N(v_p, v_z)$ is assumed to be a Maxwell–Boltzmann distribution, the imaginary part of this integral is either exactly equal to or similar to a Voigt integral depending on whether $v_z/c >$ or $< v^2/c^2$ as it is in parts of a mirror machine. For example, in a mirror machine,

$$\text{Im } \phi(\omega) \sim -\pi \frac{e^2 N_0}{m\omega^2} \frac{mc^2}{kT} \exp -\left(\frac{mc^2}{kT} \frac{\omega_0 - \omega}{\omega}\right) \quad (8)$$

for $\omega < \omega_{0\ \max}$

where $\omega_{0\ \max}$ is the resonant frequency at maximum magnetic field strength. Im $\phi(\omega) \ll 1$ for $\omega > \omega_{0\ \max}$. For a wave varying with time as exp $i\omega t$ propagated along the direction of the magnetic field, Maxwell's equations for the medium result in

$$\mathbf{P} = -\frac{c^2}{\omega^2} \frac{\partial^2 \mathbf{E}}{\partial z_2} - \mathbf{E} = \phi(\omega)\mathbf{E} \quad (9)$$

$$\mathbf{E} = \mathbf{E}_0 \exp\left[-i(\omega/c)(1+\phi)^{1/2} z + i\omega t\right]$$
$$= \mathbf{E}_0 \exp\left[i\omega(t - (\mu/c)z) - (\omega/c)kz\right] \quad (10)$$

where the index of refraction, μ, is the real part of $\sqrt{1+\phi(\omega)}$ and the absorption coefficient, k, is minus the imaginary part of $\sqrt{1+\phi(\omega)}$. For $\omega < \omega_{0\ \max}$, a wave will propagate along the axis of a mirror machine, penetrating into a region of increasing density and weaker magnetic field and therefore decreasing ω_0. At the point where $1 + \phi$ becomes negative the wave will be totally reflected. Note that if the electron velocity along the magnetic field lines is much less than the velocity perpendicular to the magnetic field lines, absorption ceases abruptly as ω becomes greater than $\omega_{0\ \max}$. This phenomenon probably accounts for Wharton's observations at Livermore of abrupt changes in emission with changes in magnetic field strength. The plasma absorbs and profusely emits all incident radiation within a frequency interval given by

$$\omega_{0\ \min}/[1 + (kT/mc^2)\log_e L] < \omega < \omega_{0\ \max} \quad (11)$$

where L is the plasma length multiplied by the coefficient of the exponential in equation (8).

Case 2. Propagation perpendicular to the magnetic field. Let us return to equation (5). In the first place one may note that radiation which is polarized in the direction of the confining field is propagated independently of the magnetic field. This simple case has been well discussed elsewhere and is of no further interest to this discussion. Assume the incident radiation to be plane polarized along the Y-axis and propagated along the X-axis. Then Maxwell's equations require that $D_x = \text{constant} = 0$, i.e. $E_x = -P_x$ and $D_y = E_y + P_y$. Substitution into equation (5) and algebraic manipulation results in the following expression for the Y-component of the polarization:

$$P_y = (\Gamma_y + \Lambda_x \Lambda_y [1 + \Gamma_x]^{-1}) E_y \equiv \Psi(\omega) E_y \quad (12)$$

where

$$\Gamma_y = \int_0^\infty \sum_{l=1}^{\infty} \frac{e^2 S_{ly} N(v_p)\, dv_p}{-m_0\omega^2(1 - v^2/c^2)^{-1/2} + ig\omega} \times \left[1 - \frac{l^2\omega_0{}^2}{(\omega[1 - v^2/c^2]^{-1/2} + 2iv_c)^2}\right]^{-1}. \quad (13)$$

Γ_x is similar and Λ_x and Λ_y have additional factors in the numerator of the integrand of $ilm_0\omega\omega_0[-m_0\omega^2(1 - v^2/c^2)^{-1/2} + ig\omega]^{-1}$. For non-relativistic energies $S_{lx} = S_{ly}$ so that $\Gamma_x = \Gamma_y$ and $\Lambda_x = \Lambda_y$. In any case the denominator, that is the quantity in brackets, resonates at

$$\frac{U}{mc^2} = \frac{l\omega_0}{\omega} \quad \text{when} \quad \omega < l\omega_0, \quad (14)$$

where U is the total energy of the electron. Again if one assumes a Maxwell–Boltzmann distribution in $v_p = v$ in a mirror machine,

$$N(v_p)\, dv_p = (N_0/kT) \exp -[(U - mc^2)/kT]\, dU$$

$$\Gamma_{ly} = -\frac{e^2 N_0}{m\omega kT} \int_{mc^2}^{\infty} \frac{S_{ly} \exp -[(U - mc^2)/kT]\, dU}{(U/mc^2)\omega} \times \frac{\left[1 - \left(\frac{l\omega_0}{\omega}\frac{mc^2}{U}\right)^2 - 4i\frac{\nu_c}{\omega}\left(\frac{l\omega_0}{\omega}\right)^2\left(\frac{mc^2}{U}\right)^3\right]}{\left[1 - \left(\frac{l\omega_0}{\omega}\frac{mc^2}{U}\right)^2\right]^2 + \left[4\frac{\nu_c}{\omega}\left(\frac{l\omega_0}{\omega}\right)^2\left(\frac{mc^2}{U}\right)^3\right]^2}. \quad (15)$$

Except for $l = 1$, the real part of this integral is quite small due to the cancellations between oscillators of different resonant frequencies. For $l = 1$ it is at least slowly varying. For a calculation of the emission or absorption it is enough to note that it is sufficiently small or at any rate sufficiently slowly varying that sharp changes do not occur within a wavelength of the radiation such that the plasma would reflect incident radiation. For $l > 1$ and $\omega > l\omega_0$, $\Gamma_{ly} \sim 0$; for $l > 1$ and $\omega < \omega_0$, Γ_{ly} is almost purely imaginary and

VOL 2 196

the same as the *quasi*-Voigt integral given in equation (8) except for a Bessel function factor:

$$\Gamma_{ly} \sim i\pi \frac{e^2 N_0}{m\omega^2} \frac{mc^2}{kT} \{J_l'(l\sqrt{1-\omega^2/l^2\omega_0^2})\} \times \exp -\left(\frac{mc^2}{kT} \frac{l\omega_0 - \omega}{\omega}\right) \quad (16)$$

Γ_{lx} is the same except for a different Bessel function factor. Λ_{lx} and Λ_{ly} are the same as Γ_{lx} and Γ_{ly} respectively except that both have an additional common factor of

$$i \frac{l\omega_0 - 2iv_c}{l\omega_0} \sim i.$$

Therefore

$$\Psi(\omega) \sim \Gamma_y - \Gamma_x\Gamma_y/[1+\Gamma_x] = \Gamma_y/[1+\Gamma_x] = \frac{(\mathrm{Re}\,\Gamma_y + i\,\mathrm{Im}\,\Gamma_y)(1+\mathrm{Re}\,\Gamma_x - i\,\mathrm{Im}\,\Gamma_x)}{(1+\mathrm{Re}\,\Gamma_x)^2 + (\mathrm{Im}\,\Gamma_x)^2}. \quad (17)$$

At non-relativistic energies $l > 1$,

$$\Gamma_{lx} \sim \Gamma_{ly} \sim i \frac{e^2 N_0}{m\omega^2} \frac{mc^2}{kT} \left[1 - \left(\frac{\omega}{l\omega_0}\right)^2\right]^{\frac{l-1}{2}} \times \exp -\left[\frac{mc^2}{kT} \frac{l\omega_0 - \omega}{\omega}\right]. \quad (18)$$

Employing Maxwell's equations, one finds an equation similar to the previous equations for propagation parallel to the field:

$$-\frac{c^2}{\omega^2} \frac{\partial^2 E_y}{\partial x^2} - E_y = \Psi(\omega) E_y \quad (19)$$

and k is the negative of the imaginary part of $\sqrt{1+\Psi(\omega)}$. At $\omega > \omega_p$, if $|\Gamma_{lx}|$ or $|\Gamma_{ly}| \gtrsim 1$, the plasma is highly absorptive and behaves like a black body. Therefore, one need only treat the case $|\Gamma_{ly}|$ and $|\Gamma_{lx}| \ll 1$. That is, $\mathrm{Im}\,\Psi(\omega) \sim \Gamma_y$ and

$$k \sim \tfrac{1}{2} i \sum_l \Gamma_{ly}, \quad (20)$$

except that the term $l = 1$ must be evaluated separately with particular attention to the real part which may be large:

$$\mathrm{Re}\,\Gamma_{1y} \sim -\frac{e^2 N_0}{m\omega^2} \frac{1}{1-(\omega_0/\omega)^2} = -\frac{\omega_p^2/4\pi}{\omega^2 - \omega_0^2} \quad (21)$$

For $\Gamma_x \sim -1$, that is near the plasma frequency, the absorptive part of $\Psi(\omega) \sim 1/\mathrm{Im}\,\Gamma(x)(\Gamma_y \sim \Gamma_x$ non-relativistic) which is >1, for example, within a narrow frequency band say $(\Delta\omega/\omega) \sim 0{\cdot}02$ for a typical Sherwood plasma. For this case the real part of $\Psi(\omega) \sim 1$.

We note that the plasma absorbs energy in a way more related to the plasma frequency than the cyclotron frequency at $\omega \lesssim \omega_p$. This is for the case when the wave is propagated perpendicular to the confining field and the longitudinal plasma oscillations are tightly coupled to the electric field of the transverse wave, and $\omega_p > \omega_0$.

For $\omega > \omega_p$, $v^2/c^2 \ll 1$, and $\omega/2\omega_0 > mc^2/kT$, an upper bound may be obtained for Γ_y. From equation (20)

$$\Gamma_y < \sqrt{2\pi} \frac{e^2 N_0}{m\omega^2} \frac{mc^2}{kT} 2\sqrt{\frac{\omega_0}{\omega}} \frac{\exp -(mc^2/kT)}{2 + (\omega/2\omega_0) - (mc^2/kT)}. \quad (22)$$

Once the absorption has been calculated the emission of the plasma is very simply estimated by multiplying the absorption by the emission of a black body at the same frequency and temperature.

I have not had time to go into more than the two simple cases of emission parallel to and perpendicular to the magnetic field. The algebra for emission in an arbitrary direction is straight forward but unwieldy and unnecessary. Changes in the refractive index have been shown to be too slight to cause internal reflection, that is, a surface emissivity of less than unity; furthermore, the emission has been shown to be independent of the plasma frequency. Therefore, the emission as a function of angle may be readily computed by integrating the emission from a single oscillator as a function of angle over all the oscillators in the plasma. It must, of course, never be estimated to be greater than black-body emission at the same temperature, angle, and frequency. We have just heard DRUMMOND give a very good demonstration of the same conclusions.

REFERENCES

BEARD D. B. (1959) Lockheed Missiles and Space Division Report LMSD 48394 (unpublished) and Supplement to this Report (1959) *Phys. Fluids* **2**, 379 **3**, 294.

BERNSTEIN I. (1958) *Phys. Rev.* **109**, 10.

DRUMMOND J. E. (1958a) *Phys. Rev.* **110**, 293; (1958b) **112**, 1460.

LANDAU L. and LIFSHITZ E. (1951) *The Classical Theory of Fields*, Addison Wesley Cambridge, Mass.

LORENTZ H. A. (1952) *The Theory of Electrons*, Chap. IV Dover Publications, New York.

SCHWINGER J. S. (1949) *Phys. Rev.* **75**, 1912.

TRUBNIKOV B. A. and KUDRYAVTSEV V. S. (1958) *Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva*, A/Conf. 15/P/2213. United Nations, N.Y. (1958),

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 98 to 108. Pergamon Press Ltd. Printed in Northern Ireland

THE CONCEPT OF CONDUCTIVITY

J. E. DRUMMOND, R. A. GERWIN and B. G. SPRINGER
Boeing Scientific Research Laboratories, Seattle, Washington

Abstract—The concept of conductivity in a hot plasma is reviewed and criticized. In particular, it is shown that even for arbitrarily small field strengths, a proportionality constant (tensor) does not in general exist between a frequency component of the field strength and the corresponding component of current density. Rather, insisting upon such a proportionality for a homogeneous plasma, defines a complete set of spatial field configurations (eigenfunctions of the conductivity). Each of these eigenfunctions separately satisfies Maxwell's equations. However, for an inhomogeneous plasma, even this is not possible since the eigenfunctions of conductivity are not separately solutions of Maxwell's equations.

It is possible, however, to define an intrinsic high-frequency property of an inhomogeneous plasma. This does not relate current density to field strength but, rather relates the field strength produced by the current density to the field strength causing the current density. Both Maxwell's equations and the particle dynamics are represented in this new property of the inhomogeneous plasma. Equating the produced and causing field strengths again defines eigenfunctions in terms of which the solution for arbitrary boundary conditions can be found. Examples will be given.

1. HISTORICAL RESUME

1.1 *Ohm's law and conductivity*

In 1827 a German physicist, reasoning by analogy with heat flow, published a speculative law for the flow of electrical current. Experts of the day hailed the concept that an electrical current existed which was proportional to the number of cells in the battery and inversely proportional to a property of the circuit called its resistance as absurd, and branded its author, George Ohm, as unworthy to teach (DUNLAP, 1944) However, over the next decade the weight of observational evidence in agreement with the law became so great and the law itself proved so convenient to use that it gained acceptance as its opponents died off. The law seemed to take on a deeper significance when it was found that the electrical resistance of a substance which depends upon its size and shape could be deduced by a quantity called its resistivity which is independent of size and shape and is a function of the material and its temperature alone.

1.2 *Frequency-dependent conductivity of a plasma*

Now let us turn to plasmas, especially cold plasmas. It has been known for several decades (APPLETON and BARNETT, 1925; ALLIS, 1956), both theoretically and experimentally, that their resistivity and its reciprocal, conductivity, depend upon the frequency of the electric field strength. Thus, Ohm's original proposal that current is proportional to voltage is not strictly true for plasmas. However, it is true that such a proportionality exists for each Fourier time component of the electric field strength, but with a different constant of proportionality for each frequency. The existence of these frequency-dependent conductivities makes the concept still a very useful one for electromagnetic problems of plasmas. The reason is that these conductivities are properties of the medium and hence a common feature of a large class of electromagnetic boundary condition problems which can be solved for once and for all. This is especially true since it is found that all of the frequency-dependent conductivities can be given by a single equation. At the same time, however, the concept of conductivity loses some of its direct intuitive appeal. One no longer connects a voltmeter and ammeter to a given simple plasma, but typically sends microwave signals through the plasma (BROWN, 1956; WHARTON, 1959). Theory is compared with experiment by deriving from the conductivity an index of refraction for the high-frequency signal.

1.3 *Form dependent conductivity of a hot plasma*

It was found several years ago that Ohm's law does not apply for metals, even for each frequency component separately (REUTER and SONDHEIMER, 1948). For metals it is not true, even for small field strengths, that the ω-component of current density at a point $\mathbf{r}$ is proportional to the ω-component of electrical field strength at $\mathbf{r}$. Instead, $\mathbf{J}_\omega(\mathbf{r})$ is given by an integral of $\mathbf{E}_\omega(\mathbf{r})$ over the neighbouring regions as shown in equation (1).

$$\mathbf{J}_\omega(\mathbf{r}) = \int \mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}_\omega(\mathbf{r}')\, d^3\mathbf{r}'. \qquad (1)$$

The reason for this is the rapid motion of the conduction electrons which can receive electric acceleration in one region and produce a current flow in another. This is also true for hot plasmas.

The kernel in equation (1) has been derived by solution of the linearized Boltzmann equation, but the form of (1) has a validity beyond that involved in any particular derivation of the kernel; being the most general homogeneous linear equation between current density and field strength. In general, of course, the kernel is a tensor operator because of the possible presence of a static magnetic field. Indeed, the magnetic field cases are usually the most interesting and important.

The possibility arises that there are certain configurations of electric field strengths for which the integral of $\mathbf{K}_\omega \cdot \mathbf{E}_\omega$ over $\mathbf{r}'$ yields just a constant tensor times $\mathbf{E}_\omega(\mathbf{r})$. For these field strengths it is true that $\mathbf{J}_\omega(\mathbf{r})$ is proportional to $\mathbf{E}_\omega(\mathbf{r})$ and the constant tensor of proportionality is the eigenvalue of the vector integral equation: if

$$\mathbf{J}_\omega(\mathbf{r}) = \boldsymbol{\sigma}_\omega \cdot \mathbf{E}_\omega(\mathbf{r}) \tag{2}$$

then

$$\boldsymbol{\sigma}_\omega \cdot \mathbf{E}_\omega(\mathbf{r}) = \int \mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}_\omega(\mathbf{r}')\, d^3\mathbf{r}'. \tag{3}$$

However, conductivity is for these situations clearly an artifice; it has not been a conclusion of the theory, but an external imposition on the theory. We insisted upon Ohm's law and found that it defined certain characteristic functions for which it was true.

Now, two questions remain: 'Are the functions so defined complete?' and 'Can they individually satisfy Maxwell's equations?' If the answer to both of these questions is 'Yes,' then our artificiality will be useful and will actually represent characteristic properties of the plasma. It turns out that for homogeneous plasmas with symmetric velocity distributions in their unperturbed state, the kernel of the integral equation is an even difference kernel. An application of Weber's integral theorems (WATSON, 1952) shows that the eigenfunctions are solutions of Helmholtz equations and hence, that they form a complete set in terms of which an arbitrary electric field is expandable Furthermore, individual members of the set satisfy Maxwell's equations. The corresponding eigenvalues (conductivity tensors) and along with them the indices of refraction, have been calculated for a number of special cases and appear in the literature (*See* MOWER, 1959).

1.4 *Inhomogeneous plasmas*

Now we turn our attention to the central problem of microwave plasma physics: the characterization of any general high-frequency properties of an inhomogeneous plasma. Our first attempt at this would quite naturally be to try to extend the concept of conductivity to inhomogeneous plasmas, but here we find that it loses much of its significance. Either we choose a convenient form for the spatial dependence of the conductivity tensor (such as a constant or as proportional to the electron density), or we simultaneously solve Maxwell's equations and the Boltzmann equation to find the form. In the first case, we find that the individual eigenfunctions of conductivity do not satisfy Maxwell's equations and, in the second case, we find that the conductivity tensors of the medium depend on conditions outside the medium: the electromagnetic boundary conditions. However, the conductivity kernel of equation (1) does still exist and is important because its symmetry bears on such important properties of the medium as the possibility of power exchange between the plasma and the electromagnetic field. The necessary and sufficient condition that the total power transfer between a plasma and the radiation field be zero for all electric fields is that the kernel be antihermitian. This statement encompasses all first-order electromagnetic interactions with quasi-steady state inhomogeneous plasmas. Furthermore, this property has been connected directly to the unperturbed state via a general derivation of the kernel. It may also be shown in general that the Fourier transform of the conduction kernel is real, and by means of causality, that this transform is one-sided. Furthermore, it can in general be shown from transform theory that if any of the orbits in the unperturbed system are periodic, resonances in the magnitude of the conduction current density occur when the period of the perturbing electric field is commensurate with the orbit period, the largest of these being for frequencies at integral multiples of the orbit frequency. The magnitude of the resonances is inversely proportional to the momentum transfer collision frequency for small ratios of this frequency to the resonance frequency. Examples of this are the well-known cyclotron resonance and the less well-known high-frequency conduction resonance for plasmas having quasi-static electric potential maxima as with ion plasma waves. These, and other theorems are proved in the appendices of this paper. The conduction kernels for examples of a warm inhomogeneous plasma and some of its limiting cases are also calculated. For certain situations, these kernels are approximately antihermitian.

2. THE JOINT KINETIC-ELECTROMAGNETIC OPERATOR

The trouble with the current density operator derived from the Boltzmann equation for inhomogeneous

plasmas is that its eigenfunctions differ from those for the current density operator given by Maxwell's equations. This suggests that we combine the operators into a single kinetic-electromagnetic operator which expresses at once the kinetics, dynamics, electromagnetics and first order correlation statistics for inhomogeneous plasmas. This can be done most compactly by expressing Maxwell's equations as an integral for the electric field strength in terms of the conduction current density and then substituting for this current density its expression from the conduction equation just considered. The result is a single inhomogeneous vector integral equation for the electric field strength.

$$\mathbf{E}_\omega(\mathbf{r}) = \mathbf{F}_\omega(\mathbf{r}) + \lambda \int K_\omega(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}_\omega(\mathbf{r}')\, d^3\mathbf{r}' \qquad (4)$$

The kernel of this integral equation is a true property of the medium and characterizes all its dynamics and electromagnetics which are independent of the electromagnetic boundary conditions. The boundary conditions are included by the inhomogeneous term $\mathbf{F}_\omega$. This characterization can be put in other forms. For instance, the kernel may define a complete set of eigenfunctions and corresponding eigenvalues. The eigenfunctions are orthogonal. If the kernel is non-hermitian, it can be made hermitian by convolution with itself on the left and also by convolution on the right. These two different hermitian kernels then define two orthogonal sets of eigenfunctions 'with common eigenvalues' and the original kernel can be expanded in the biorthogonal series of eigenfunctions divided by the common eigenvalue (TRICOMI, 1957). It may be that the two orthogonal sets of functions are not complete. In case the sets of orthogonal functions are incomplete, it might be possible to choose boundary conditions so that $\mathbf{F}_\omega(\mathbf{r}')$ is orthogonal to $\mathbf{K}_\omega(\mathbf{r},\mathbf{r}')$ in which case the plasma would be transparent to such waves.

3. SUMMARY OF THE NEW WORK

Now let us review what has been done: (i) A class of high-frequency plasma perturbation problems which can be described by the Boltzmann equation below has been considered.

$$\frac{\partial f}{\partial t} + \mathbf{V} \cdot \nabla_{\mathbf{r}} f + \frac{e}{m}\left[\mathbf{E}(\mathbf{r}, t) + \frac{\mathbf{V}}{c} \times \mathbf{B}(\mathbf{r}, t)\right] \cdot \nabla_{\mathbf{V}} f = -\nu(f - f_0)$$

$$\mathbf{V} \cdot \nabla_{\mathbf{r}} f_0 + \frac{e}{m}\left[\mathbf{E}_0(\mathbf{r}) + \frac{\mathbf{V}}{c} \times \mathbf{B}_0(\mathbf{r})\right] \cdot \nabla_{\mathbf{V}} f_0 = 0 \qquad (5)$$

$$f = f_0 + f_1 \qquad f_1 \ll f_0.$$

Notice that a general time and space dependent magnetic as well as electric field is allowed for. It should also be noted that the quasi-static state about which the perturbations occur is allowed to be inhomogeneous and consistent with spatially dependent electric and magnetic fields. The Boltzmann equations written here do not allow for the step function potentials representing particles colliding with walls. Consequently, this analysis will be valid to within either about a mean free path of a wall or about a mean cyclotron orbit diameter of a wall. The Boltzmann equations shown here, together with Maxwell's equations, are the conditions under which the general theory is developed. They allow a wide variety of special cases. (ii) The general solution to the perturbation Boltzmann equation was obtained and from this the current and charge densities were calculated. Thus, an explicit, general formula has been derived for the conduction kernel. This has been done earlier (REUTER and SONDHEIMER, 1948). (iii) Maxwell's equations were solved for $\mathbf{E}_\omega$ as a functional of $\mathbf{J}_\omega$ by means of the usual Green's function for a Helmholtz equation. Into this equation was substituted the functional for $\mathbf{E}_\omega$ previously derived from the Boltzmann equation. The result is a single vector integral equation for $\mathbf{E}_\omega$ with an explicit formula for the kernel covering all the special cases allowed for in the general formulation. Thus, we have developed a methodology for dealing with perturbations of a rather general class of quasi-static inhomogeneous plasmas but have only scratched the surface in applying it to the wide range of problems of interest. However, let us consider some of these simple cases.

4. EXAMPLES

4.1 *Warm homogeneous plasma*

Equation (6) shows the asymptotic form of the conduction kernel for a warm homogeneous plasma.

$$\mathbf{K}(\mathbf{r}, \mathbf{r}') = \frac{-2\omega\left(i - \dfrac{\nu}{\omega}\right)}{\sqrt{3}\,\lambda_D^2} \frac{(\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

$$\times \exp\left\{-\frac{3}{2}\left[\frac{\omega}{\omega_p}\frac{|\mathbf{r} - \mathbf{r}'|}{\lambda_D}\left(i - \frac{\nu}{\omega}\right)\right]^{2/3}\right\} \frac{|\mathbf{r} - \mathbf{r}'|^2}{\lambda_D^2} \gg 1$$

$$\lambda_D = \sqrt{\frac{KT}{4\pi n e^2}} = \sqrt{\frac{KT}{m\omega_p^2}}. \qquad (6)$$

Notice the interesting exponential dependence on the two-thirds power of the ratio of $|\mathbf{r} - \mathbf{r}'|$ to the Debye length, λ_D. It is reminiscent of the asymptotic form of the longitudinal electric field penetrating into a plasma from a plane electrode (LANDAU, 1946). The present kernel should lead to that result, if someone

would care to solve the appropriate Weiner-Hopf integral equation that the regenerative equation becomes. It should also lead to considerably more than that. Not only does it include collisions, but it also contains the influence of oscillating transverse electric and magnetic fields. Another important thing to note about this kernel is that it is approximately antihermitian. From the previously stated results, then, we know that there will be very little power transfer between the radiation field and the plasma electrons. Another item of interest about the kernel is that it is a symmetric difference kernel. This immediately leads to the conclusion that its eigenfunctions are solutions to the Helmholtz equation, thus agreeing with earlier work, and yielding the appropriate conductivity tensors from the conduction equation.

4.2 *Inhomogeneous equilibrium plasma*

Now let us suppose that changes in an externally applied electric field occur so slowly that collisions of all kinds are able to keep the quasi-static electron distribution in equilibrium as shown by the Boltzmann factor.

$$f_0 = C \exp\left\{-\frac{\frac{1}{2}mv^2 + e\phi}{KT}\right\}$$

$$\phi = \mathbf{E}_0 \cdot \mathbf{r} \tag{7}$$

Let us also suppose that the ion currents that result from the field just cancel the Maxwellian displacement currents of the field itself, as for instance in ion plasma oscillations as found by TONKS and LANGMUIR (1929). If the electric field rises steadily, the TONKS and LANGMUIR formula shows that it will be independent of position. This gives an exponential density distribution as shown in Fig. 1.

It is this simple inhomogeneous state that our next example concerns. An explicit formula has been derived for the conduction kernel which is presented in the appendix of this paper. However, the criterion for no power transfer can be applied immediately by inspection of the form of the derived kernel. The conclusion is that no total power transfer can occur for this case without collisions. This is contrary to what one usually believes about inhomogeneities coupling a plasma to the radiation field. However, it is in agreement with the observations of ASH and GABOR (1955) on the non-radiation of plasma oscillations in a sheath. Furthermore, it illustrates the usefulness of the theorem. The solution to no difficult electromagnetic boundary condition problem was required.

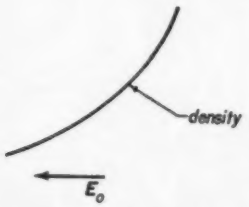


FIG. 1.—Special case of inhomogeneous plasma with quasi-static electric field.

A simple inspection of the conduction kernel showed that coherent radiation will not occur for the state considered. This would appear to be useful in the design of experiments to detect plasma oscillations without probe disturbance, or in the converse problem of inducing plasma oscillations by externally produced radiation. Once it has been determined that an initial state allows an interesting coupling to the radiation field, the detailed solution to the simultaneous kinetic and electromagnetic problem has been reduced to the solution of a single vector integral equation whose kernel has been derived in general terms for an inhomogeneous plasma.

APPENDICES

A1. THE CONDUCTION KERNEL

A1.1 *Derivation*

Consider situations which can be represented by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m}\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{H}\right] \cdot \nabla_{\mathbf{v}} f = -\nu(f - f_0) \tag{1}$$

where the steady state f_0 satisfies

$$\mathbf{v} \cdot \nabla f_0 + \frac{e}{m}\left[\mathbf{E}_0 + \frac{\mathbf{v}}{c} \times \mathbf{H}_0\right] \cdot \nabla_{\mathbf{v}} f_0 = 0 \tag{2}$$

and $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell's equations with charge density given by

$$\rho = e\int_\infty f\, d^3\mathbf{v} \tag{3}$$

and current density given by

$$\mathbf{J} = e\int_\infty f\mathbf{v}\, d^3\mathbf{v}. \tag{4}$$

Regarding $f_1 \equiv f - f_0$, $\mathbf{E}_1 \equiv \mathbf{E} - \mathbf{E}_0$, and $\mathbf{H}_1 \equiv \mathbf{H} - \mathbf{H}_0$ as of first order in a perturbation, and dropping second order quantities, one obtains

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \nabla f_1 + \frac{e}{m}\left[\mathbf{E}_0 + \frac{\mathbf{v}}{c} \times \mathbf{H}_0\right] \cdot \nabla_{\mathbf{v}} f_1 = \frac{-e}{m}\left[\mathbf{E}_1 + \frac{\mathbf{v}}{c} \times \mathbf{H}_1\right] \cdot \nabla_{\mathbf{v}} f_0 - \nu f_1 \tag{5}$$

This equation can be solved by the method of characteristics (KELLS, 1947) with result

$$f_1(\mathbf{r}, \mathbf{v}, t) = \frac{-e}{m}\int_0^\infty \exp(-\nu s)\left[\mathbf{E}_1(\mathbf{r}', t') + \frac{\mathbf{v}'}{c} \times \mathbf{H}_1(\mathbf{r}', t')\right] \cdot \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v}')\, ds \quad (6)$$

where

$$t(s') \equiv t - s \quad (7)$$

$$\mathbf{v}'(s) \equiv \mathbf{v} - \frac{e}{m}\int_0^s \left[\mathbf{E}_0(\mathbf{r}'(\sigma)) + \frac{\mathbf{v}'(\sigma)}{c} \times \mathbf{H}_0(\mathbf{r}'(\sigma))\right] d\sigma \quad (8)$$

$$\mathbf{r}'(s) \equiv \mathbf{r} - \int_0^s \mathbf{v}'(\sigma)\, d\sigma \quad (9)$$

This, together with equation (4) gives for the Fourier components of current density

$$\mathbf{J}_\omega(\mathbf{r}) = \frac{-e^2}{m}\int_\infty \mathbf{v}\, d^3\mathbf{v} \int_0^\infty \exp(i\omega - \nu)s \times \left[\mathbf{E}_\omega(\mathbf{r}') - \frac{i}{\omega}\mathbf{v}' \times \nabla_{\mathbf{r}'}\,|\, \times \mathbf{E}_\omega(\mathbf{r}')\right] \cdot \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v}')\, ds \quad (10)$$

where

$$\mathbf{J}_\omega(\mathbf{r}) = \frac{1}{\sqrt{2\pi}}\int \exp(i\omega t)\mathbf{J}(\mathbf{r}, t)\, dt, \quad \text{etc.} \quad (11)$$

The $\nabla_{\mathbf{r}'} \times \mathbf{E}_\omega(\mathbf{r}')$ term under the integrals in equation (10) may be rewritten as

$$\frac{i}{\omega}[\mathbf{v}' \times \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v}')] \cdot \nabla_{\mathbf{r}'} \times \mathbf{E}_\omega(\mathbf{r}') = \frac{i}{\omega}(\nabla_{\mathbf{r}'} \times [\mathbf{v}' \times \underset{\mathbf{v}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v}')]) \cdot \mathbf{E}_\omega(\mathbf{r}') + \frac{i}{\omega}\nabla_{\mathbf{r}'} \cdot (\mathbf{E}_\omega(\mathbf{r}') \times [\mathbf{v}' \times \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v}')]) \quad (12)$$

Equations (7–9) of the unperturbed problem may be assumed solved simultaneously with Maxwell's equations. Thus the initial and final velocities v' and v may be expressed in terms of the initial and final positions, r' and r, and the duration, s. Using these variables and the expression (12) one obtains from equation (10)

$$\mathbf{J}_\omega(\mathbf{r}) = \frac{-e^2}{m}\int_0^\infty \exp[(i\omega - \nu)s]\, ds \times \int_\infty \{\mathbf{v}(\mathbf{r}, \mathbf{r}', s)\mathbf{F}_1(\mathbf{r}, \mathbf{r}', s) \cdot \mathbf{E}_\omega(\mathbf{r}') + \mathbf{v}(\mathbf{r}, \mathbf{r}', s) \times \underset{\mathbf{r},s=\text{const.}}{\nabla_{\mathbf{r}'}} \cdot [\mathbf{E}_\omega(\mathbf{r}') \times \mathbf{F}_2(\mathbf{r}, \mathbf{r}', s)]\}\, |\mathscr{J}(\mathbf{v}/\mathbf{r}')|\, d^3\mathbf{r}' \quad (13)$$

where

$$\mathbf{F}_1(\mathbf{r}, \mathbf{r}', s) \equiv \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v}') \frac{i}{\omega} \underset{\mathbf{r},s=\text{const.}}{\nabla_{\mathbf{r}'}}\,| \times [\mathbf{v}' \times \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}}\,| f_0(\mathbf{r}', \mathbf{v}')] \quad (14)$$

$$\mathbf{F}_2(\mathbf{r}, \mathbf{r}', s) \equiv -\frac{i}{\omega}\mathbf{v}' \times \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v}') \quad (15)$$

and $\mathscr{J}(\mathbf{v}/\mathbf{r}')$ is the Jacobian of the transformation from $\mathbf{v}$ to $\mathbf{r}'$.

$$\mathscr{J}(\mathbf{v}/\mathbf{r}') = \begin{vmatrix} \frac{\partial v_x}{\partial x'} & \frac{\partial v_y}{\partial x'} & \frac{\partial v_z}{\partial x'} \\ \frac{\partial v_x}{\partial y'} & \frac{\partial v_y}{\partial y'} & \frac{\partial v_z}{\partial y'} \\ \frac{\partial v_x}{\partial z'} & \frac{\partial v_y}{\partial z'} & \frac{\partial v_z}{\partial z'} \end{vmatrix} S(\mathbf{r}, \mathbf{r}') \quad (16)$$

where the 'shadow' function, $S(\mathbf{r}, \mathbf{r}')$ is introduced to account for multiply connected plasmas such as a plasma around a probe. $\mathscr{J}$ is the velocity-configuration space volume ratio, $\frac{d^3\mathbf{v}}{d^3\mathbf{r}'}$, and in such plasmas there may be regions of space $d^3\mathbf{r}'$ for which there exists no corresponding $d^3\mathbf{v}$ due to the trajectories being intercepted by the probe. For these values of $\mathbf{r}'$, $S(\mathbf{r}, \mathbf{r}') = 0$, and is otherwise 1. This effect could possibly have been included from the start by introducing a 'sink' term in the Boltzmann equation.

In equation (13) the term involving $\underset{\mathbf{v}'=\text{const.}}{\nabla_{\mathbf{r}'}} \cdot$ can be integrated by parts on $\mathbf{r}'$. The integrated part vanishes because $\mathbf{v}'(\mathbf{r}, \mathbf{r}', s) \xrightarrow[\mathbf{r}'\to\infty]{} \infty$ and $\mathbf{v}'^n f_0(\mathbf{r}', \mathbf{v}') \xrightarrow[\mathbf{v}'\to\infty]{} 0$. The unintegrated part can be written

$$-\int_\infty [\mathbf{E}_\omega(\mathbf{r}') \times \mathbf{F}_2(\mathbf{r}, \mathbf{r}', s)] \cdot \underset{\mathbf{r},s=\text{const.}}{\nabla_{\mathbf{r}'}} [\mathbf{v}(\mathbf{r}, \mathbf{r}', s)\,|\mathscr{J}(\mathbf{v}/\mathbf{r}')|]\, d^3\mathbf{r}' \quad (17)$$

$$= \frac{i}{\omega}\int_\infty ([\mathbf{E}_\omega(\mathbf{r}') \cdot \mathbf{v}'][\underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v})] \cdot \underset{\mathbf{r},s=\text{const.}}{\nabla_{\mathbf{r}'}} \mathbf{v}\,|\mathscr{J}(\mathbf{v}/\mathbf{r}')| - [\mathbf{E}_\omega(\mathbf{r}') \cdot \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v}')]\mathbf{v}' \cdot \underset{\mathbf{r},s=\text{const.}}{\nabla_{\mathbf{r}'}} [\mathbf{v}\,|\mathscr{J}(\mathbf{v}/\mathbf{r}')|])\, d^3\mathbf{r}' \quad (18)$$

Combining equations (13) and (18) one obtains

$$\mathbf{J}_\omega(\mathbf{r}) = \frac{-e^2}{m}\int_0^\infty \exp[(i\omega - \nu)s]\, ds \times \int \mathscr{H}_\omega'(\mathbf{r}, \mathbf{r}', s) \cdot \mathbf{E}_\omega(\mathbf{r}')\, d^3\mathbf{r}' \quad (19)$$

where

$$\mathscr{H}_\omega'(\mathbf{r}, \mathbf{r}', s) \equiv |\mathscr{J}|\, \mathbf{v}\mathbf{F}_1 - \frac{i}{\omega}[\underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v}') \cdot \underset{\mathbf{r},s=\text{const.}}{\nabla_{\mathbf{r}'}} \mathbf{v}\,|\mathscr{J}|]\,\mathbf{v}' - \frac{i}{\omega}[\mathbf{v}' \cdot \underset{\mathbf{r},s=\text{const.}}{\nabla_{\mathbf{r}'}} \mathbf{v}\,|\mathscr{J}|][\underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'}} f_0(\mathbf{r}', \mathbf{v}')] \quad (20)$$

VOL. 2 196

Finally integrating† with respect to s one obtains

$$\mathbf{J}_\omega(\mathbf{r}) = \int_\infty \mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}_\omega(\mathbf{r}')\, d^3\mathbf{r}' \tag{21}$$

where

$$\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') \equiv \frac{-e^2}{m} \int_0^\infty \exp(i\omega - \nu)s \mathscr{H}_\omega'(\mathbf{r}, \mathbf{r}', s)\, ds \tag{22}$$

$$= \frac{-e^2}{m} \int_0^\infty \exp(i\omega - \nu)s \mathscr{H}_1(\mathbf{r}, \mathbf{r}', s)\, ds \tag{23}$$

$$\frac{-ie^2}{\omega m} \int_0^\infty \exp(i\omega - \nu)s \mathscr{H}_2(\mathbf{r}, \mathbf{r}', s)\, ds$$

where

$$\mathscr{H}_1(\mathbf{r}, \mathbf{r}', s) \equiv |\mathscr{J}| \underset{\mathbf{r}'=\text{const.}}{\mathbf{v}\nabla_{\mathbf{v}'} f_0(\mathbf{r}', \mathbf{v}')}, \tag{24}$$

$$\mathscr{H}_2(\mathbf{r}, \mathbf{r}', s) = |\mathscr{J}| \underset{\mathbf{r},s=\text{const.}}{\mathbf{v}\nabla_{\mathbf{r}'}} \times [\mathbf{v}' \times \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'} f_0(\mathbf{r}', \mathbf{v}')}]$$
$$+ [\underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'} f_0(\mathbf{r}', \mathbf{v}')} \cdot \underset{\mathbf{r},s=\text{const.}}{\nabla_{\mathbf{r}'} \mathbf{v}} |\mathscr{J}|]\mathbf{v}' \tag{25}$$
$$- [\mathbf{v}' \cdot \underset{\mathbf{r},s=\text{const.}}{\nabla_{\mathbf{r}} \mathbf{v}} |\mathscr{J}|][\underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'} f_0(\mathbf{r}', \mathbf{v}')}]$$

are real and independent of ω. Integrating the second integral in equation (23) by parts, one obtains

$$\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') = \frac{-e^2}{m} \int_0^\infty \exp[(i\omega - \nu)s] \mathscr{H}(\mathbf{r}, \mathbf{r}', s)\, ds \tag{26}$$

where

$$\mathscr{H}(\mathbf{r}, \mathbf{r}', s) = \mathscr{H}_1(\mathbf{r}, \mathbf{r}', s)$$
$$- \int_0^s \exp[\nu(s - s')] \mathscr{H}_2(\mathbf{r}, \mathbf{r}', s')\, ds' \tag{27}$$

is real and independent of ω. The integrated part vanishes as can be seen by writing the second integral as

$$-\frac{ie^2}{m\omega} \int_0^\infty \exp[(i\omega - \nu)s] \mathscr{H}_2(\mathbf{r}, \mathbf{r}', s)\, ds$$
$$= -\frac{ie^2}{m\omega} \lim_{\alpha \to 0} \int_0^\infty \exp[(i\omega - \alpha)s] \exp(-\nu s) \mathscr{H}_2(\mathbf{r}, \mathbf{r}', s)\, ds$$

The integrated part is then

$$-\frac{ie^2}{m\omega} \lim_{\alpha \to 0} \left(\left[\exp(i\omega - \alpha)s \right.\right.$$
$$\left.\left. \times \int_0^s \exp(-\nu s') \mathscr{H}_2(\mathbf{r}, \mathbf{r}', s')\, ds' \right]_0^\infty \right)$$

which evidently vanishes at both $s = 0$ and $s = \infty$.

Such a reduction has to be possible since $\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}')$ gives the relation between two Fourier transforms $\mathbf{E}_\omega(\mathbf{r}')$ and $\mathbf{J}_\omega(\mathbf{r})$ of real functions $\mathbf{E}(\mathbf{r}', t)$ and $\mathbf{J}(\mathbf{r}, t)$.

† Note that it is not always possible to invert the order of integration over s and $\mathbf{r}'$ and when this is not possible the kernel may be defined in this manner only symbolically.

In the same way, one obtains

$$\rho_\omega(\mathbf{r}) = \int_\infty \mathbf{k}_\omega(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}_\omega(\mathbf{r}')\, d^3\mathbf{r}' \tag{28}$$

where

$$\mathbf{k}_\omega(\mathbf{r}, \mathbf{r}') = \frac{-e^2}{m} \int_0^\infty \exp[(i\omega - \nu)s]\, \mathscr{L}(\mathbf{r}, \mathbf{r}', s)\, ds \tag{29}$$

with

$$\mathscr{L}(\mathbf{r}, \mathbf{r}', s) = \mathscr{L}_1(\mathbf{r}, \mathbf{r}', s)$$
$$+ \int_0^s \exp[\nu(s - s')] \mathscr{L}_2(\mathbf{r}, \mathbf{r}', s')\, ds'$$

where

$$\mathscr{L}_1(\mathbf{r}, \mathbf{r}', s) = |\mathscr{J}| \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'} f_0(\mathbf{r}', \mathbf{v}')}$$

$$\mathscr{L}_2(\mathbf{r}, \mathbf{r}', s) = |\mathscr{J}| \underset{\mathbf{v}'=\text{const.}}{\nabla_{\mathbf{r}'}} \times [\mathbf{v}' \times \underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'} f_0(\mathbf{r}', \mathbf{v}')}]$$
$$+ [\underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'} f_0(\mathbf{r}', \mathbf{v}')} \cdot \underset{\mathbf{v}'=\text{const.}}{\nabla_{\mathbf{r}'} \mathscr{J}}]\mathbf{v}'$$
$$- [\mathbf{v}' \cdot \underset{\mathbf{v}'=\text{const.}}{\nabla_{\mathbf{r}'}} |\mathscr{J}|][\underset{\mathbf{r}'=\text{const.}}{\nabla_{\mathbf{v}'} f_0(\mathbf{r}', \mathbf{v}')}]$$

It is possible that the inhomogeneity of the plasma may be of such a form that the collision frequency will depend on position and time. The above equations become more complicated but the conductivity kernel may still be derived in the same form as above.

A1.2 *Inversion*

The theory of vector integral equations (GERWIN, DRUMMOND and SPRINGER, 1960) gives the following inversion formula for an integral equation of the first kind as equation (A1.1–21).

$$\mathbf{E}_\omega(\mathbf{r}) = \mathbf{\Gamma}(\mathbf{r}) + \sum_{j=1}^\infty \lambda_j \mathbf{E}_j(\mathbf{r})$$
$$\times \int_R \int_R \mathbf{E}_j^*(\mathbf{r}'') \cdot \mathbf{K}^+(\mathbf{r}'', \mathbf{r}') \cdot \mathbf{J}_\omega(\mathbf{r}')\, d^3\mathbf{r}' \tag{1}$$

when it converges, with λ_j and $\mathbf{E}_j(\mathbf{r})$ respectively the eigenvalues and orthonormalized eigenfunctions of the Hermitian kernel

$$\mathbf{H}(\mathbf{r}, \mathbf{r}') = \int_R \mathbf{K}^+(\mathbf{r}'', \mathbf{r}) \cdot \mathbf{K}(\mathbf{r}'', \mathbf{r}')\, d^3\mathbf{r}'' = \mathbf{H}^+(\mathbf{r}', \mathbf{r}) \tag{2}$$

where

$$\mathbf{K}_{ij}^+(\mathbf{r}'', \mathbf{r}) \equiv \mathbf{K}_{ji}^*(\mathbf{r}'', \mathbf{r}) \tag{3}$$

The function $\mathbf{\Gamma}(\mathbf{r})$ is orthogonal to all the eigenfunctions, $\mathbf{E}_j(\mathbf{r})$, $j = 1, 2, 3, \ldots$ but is otherwise arbitrary. If the set of eigenfunctions generated by the kernel $\mathbf{H}$ is complete,

$$\mathbf{\Gamma}(\mathbf{r}) \equiv 0 \tag{4}$$

as will be assumed. This is reasonable on the general

* Complex conjugate.

physical ground of determinancy and can be shown for kernels that occur in particular plasma problems.

If the conduction kernel is either Hermitian or antihermitian, the inversion formula is simpler:

$$\mathbf{E}_\omega(\mathbf{r}) = \sum_{j=1}^{\infty} \lambda_j \mathbf{E}_j(\mathbf{r}) \int_R \mathbf{E}_j^*(\mathbf{r}') \cdot \mathbf{J}_\omega(\mathbf{r}')\, d^3\mathbf{r}' \qquad (5)$$

with λ_j the $\binom{\text{real}}{\text{imaginary}}$ eigenvalues and $\mathbf{E}_j(\mathbf{r})$ the orthonormalized eigenfunctions of the $\binom{\text{Hermitian}}{\text{antihermitian}}$ kernel $\mathbf{K}(\mathbf{r}, \mathbf{r}')$. (An antihermitian kernel can be made Hermitian by multiplying it by i). The function $\boldsymbol{\Gamma}(\mathbf{r})$ has again been set equal to zero on the assumption that the $\mathbf{E}_j(\mathbf{r})$ are complete.

A1.3 *Condition for no power exchange*

The d.c. component of the power density (divergence of the Poynting vector) arising from electromagnetic fields of frequency $|\omega|$ is

$$P_\omega(\mathbf{r}) = \operatorname{Re} \mathbf{E}_\omega^*(\mathbf{r}) \cdot \mathbf{J}_\omega(\mathbf{r}) \qquad (1)$$

The total power throughout the entire inhomogeneous plasma is (using equation (A1.1–21))

$$\mathscr{P}_\omega = \operatorname{Re} \int_\infty \mathbf{E}_\omega^*(\mathbf{r}) \cdot \mathbf{J}_\omega(\mathbf{r})\, d^3\mathbf{r} \qquad (2)$$

$$= \operatorname{Re} \int_\infty \int_\infty \mathbf{E}_\omega^*(\mathbf{r}) \cdot \mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}_\omega(\mathbf{r}')\, d^3\mathbf{r}\, d^3\mathbf{r}' \qquad (3)$$

$$= \operatorname{Re} \int_\infty \int_\infty \mathbf{E}_\omega(\mathbf{r}') \cdot \mathbf{K}_\omega^*(\mathbf{r}', \mathbf{r}) \cdot \mathbf{E}_\omega^*(\mathbf{r})\, d^3\mathbf{r}\, d^3\mathbf{r}' \qquad (4)$$

If $\mathbf{A} \cdot \mathbf{K}_\omega^*(\mathbf{r}', \mathbf{r}) \equiv -\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') \cdot \mathbf{A}$ for any $\mathbf{A}$ (i.e. if $\mathbf{K}_\omega$ is antihermitian), then

$$\mathscr{P}_\omega = -\operatorname{Re} \int_\infty \int_\infty \mathbf{E}_\omega^*(\mathbf{r}) \cdot \mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') \cdot \mathbf{E}_\omega(\mathbf{r}')\, d^3\mathbf{r}\, d^3\mathbf{r}'$$

$$= -\mathscr{P}_\omega$$

or

$$\mathscr{P}_\omega = 0.$$

Thus if $\mathbf{K}_\omega$ is antihermitian, it will be impossible for electromagnetic power to be absorbed or radiated by the mechanisms allowed for by the linearized Boltzmann equation for an inhomogeneous plasma solved in the self-consistent field approximation. On the other hand, if the total power is zero for arbitrary† electric field configurations:

$$0 = \iint \mathbf{E}_\omega^*(\mathbf{r}) \cdot [\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') + \mathbf{K}_\omega^+(\mathbf{r}', \mathbf{r})] \cdot \mathbf{E}_\omega(\mathbf{r}')\, d^3\mathbf{r}\, d^3\mathbf{r}' \qquad (5)$$

† $\mathbf{E}_\omega$ only need satisfy Maxwell's equations with *arbitrary* current distributions since there may be other currents than those induced in the plasma (such as for example, those produced by beams or by a modulated density of β-emitters in a nuclear reactor). By using envelopes of vanishingly small currents we can produce any arbitrary $\mathbf{E}_\omega$ without affecting the conductivity kernel.

where $+$ indicates the Hermitian conjugate, then

$$\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') \equiv -\mathbf{K}^+(\mathbf{r}', \mathbf{r}). \qquad (6)$$

Therefore, the necessary and sufficient condition that the total power transfer vanish identically is that $\mathbf{K}$ be antihermitian. It thus becomes important to discover the conditions under which $\mathbf{K}_\omega$ will be antihermitian.

It would be more useful if we could relate the power transfer to the transform kernel $\mathscr{H}$ since this is the kernel deduced directly from the Boltzmann equation. This can be done by means of the inversion of the one-sided Fourier transform, equation (A1.1–26);

$$\exp(-\nu s)\mathscr{H}(\mathbf{r}, \mathbf{r}', s) = \frac{-m}{2\pi e^2} \times \int_{-\infty+i\gamma}^{+\infty+i\gamma} \exp(-i\omega s)\, \mathbf{K}_\omega(\mathbf{r}, \mathbf{r}')\, d\omega \text{ for } s \geq 0 \qquad (7)$$

where $\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}')$ is analytic for $\operatorname{Im} \omega \geqslant \gamma$. From equations (A1.1–24)–(A1.1–27),

$$\gamma = 0 \text{ if } \nu = 0 \text{ and } \mathscr{H}(s) \xrightarrow[s \to \infty]{} 0. \qquad (8)$$

Taking the Hermitian adjoint of each side and interchanging $\mathbf{r}$ and $\mathbf{r}'$ we have

$$\exp(-\nu s)\mathscr{H}^+(\mathbf{r}', \mathbf{r}, s) = \frac{-m}{2\pi e^2} \int_{-\infty-i\gamma}^{\infty-i\gamma} \exp(i\omega s)\mathbf{K}_\omega^+(\mathbf{r}', \mathbf{r})\, d\omega \qquad (9)$$

If $\nu = \gamma = 0$, then equation (9) becomes an invertible Fourier transform and thus the necessary condition that $\mathbf{K}_\omega^+(\mathbf{r}', \mathbf{r}) = -\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}')$ is seen to be

$$\mathscr{H}^+(\mathbf{r}', \mathbf{r}, s) = -\mathscr{H}(\mathbf{r}, \mathbf{r}', -s) \qquad (10)$$

where $\mathscr{H}(\mathbf{r}, \mathbf{r}', -s)$ is the analytic continuation of $\mathscr{H}(\mathbf{r}, \mathbf{r}', s)$ and is given by

$$\mathscr{H}(\mathbf{r}, \mathbf{r}', -s) \equiv \frac{-m}{2\pi e^2} \times \int_{-\infty}^{\infty} \exp(+i\omega s)\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}')\, d\omega \text{ for } s \geqslant 0.$$

Thus the necessary and sufficient condition that the total power transfer between a collisionless plasma and the electromagnetic field be zero (under the conditions of the present analysis) is that the Fourier transform of the conduction kernel satisfy equation (10).

A1.4 *Resonances*

It is interesting to note that the Laplace transform (A1.1–26) can be written

$$\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') = [1 - \exp(-\nu\tau)]^{-1} \times \int_0^\tau \exp[(i\omega - \nu)s]\overline{\mathscr{H}}(\mathbf{r}, \mathbf{r}', s)\, ds \qquad (1)$$

if $\exp(i\omega s)\mathscr{H}(\mathbf{r}, \mathbf{r}', s)$ is periodic with period τ. This will be so if $\mathscr{H}$ has a period p, which is a rational multiple, $\frac{n}{m}$ of $\frac{2\pi}{\omega}$ where n and m are relatively prime integers. The period τ will be

$$\tau = \frac{2\pi n}{\omega} = mp. \tag{2}$$

The resonances are likely to be large if $\nu\tau \ll 1$. Then

$$\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}')_{\text{res.}} \propto \frac{1}{\nu\tau}. \tag{3}$$

Of these resonances, the largest will be obtained when m and n are the smallest, i.e. when

$$\omega = \frac{2\pi}{p}. \tag{4}$$

If the unperturbed state has a period T in the motion of individual electrons, it can be seen from equations (A1.1–24) and (A1.1–25) that $\mathscr{H}_1$ and $\mathscr{H}_2$ will be periodic in s with period T; but because of the multiplication of several periodic functions of this period, it may also be periodic with a sub-multiple of T. Thus from equations (A1.1–23)

$$p = \frac{T}{k}, \qquad k = 1, 2, 3, \ldots$$

or from equation (4)

$$\omega = 2\pi k/T.$$

Thus the principal resonances in $\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}')$ occur at multiples of the frequency of the orbital motions in the unperturbed state.

A1.5 *Conduction kernel for warm homogeneous plasma* ($\mathbf{H}_0 = 0$)

As a useful limiting case, consider

$$f_0 = n_0(m/2\pi KT)^{3/2} \exp(-m\mathbf{v}^2/2KT) \tag{1}$$

$$\mathbf{H}_0 = 0. \tag{2}$$

From equation (A1.1–2) this means also that

$$\mathbf{E}_0 = 0. \tag{3}$$

Now equations (A1.1–8) and (A1.1–9) give

$$\mathbf{v}'(s) = \mathbf{v} \tag{4}$$

$$\mathbf{r}'(s) = \mathbf{r} - \mathbf{v}s \quad \text{or} \quad \mathbf{v} = \frac{\mathbf{r} - \mathbf{r}'}{s}. \tag{5}$$

From equation (5), equation (A1.1–16) becomes

$$\mathscr{J} = \frac{-1}{s^3}. \tag{6}$$

Equation (1) gives

$$\nabla_{\mathbf{v}} f_0(\mathbf{r}', \mathbf{v}') = \frac{-n_0 m}{KT}(m/2\pi KT)^{3/2}\frac{\mathbf{r} - \mathbf{r}'}{s} \times \exp\left[-m\left(\frac{\mathbf{r} - \mathbf{r}'}{s}\right)^2 \Big/ 2KT\right] \tag{7}$$

Thus the transform kernel given by equation (A1.1–24) through (A1.1–27) is

$$\mathscr{H}(\mathbf{r}, \mathbf{r}', s) = \frac{-n_0 m}{KTs^5}(m/2\pi KT)^{3/2} \times \exp\left[-m\left(\frac{|\mathbf{r} - \mathbf{r}'|}{s}\right)^2 \Big/ 2KT\right](\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}') \tag{8}$$

By inspection of equation (A1.1–25), it can be seen that $\mathscr{H}_2$ is identically zero. This will be true whenever f_0 depends upon $\mathbf{v}$ only through $|\mathbf{v}|^2$.

The conduction kernel is thus given by

$$\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') = \frac{n_0 e^2 m^{3/2}(\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}')}{KT(2\pi KT)^{3/2}} \times \int_0^\infty \exp\left[\frac{m\omega^2 |\tau - \tau'|^2}{2KT} h(s)\right]\frac{ds}{s^5} \tag{9}$$

where

$$h(s) \equiv \frac{2KT(i - \nu/\omega)s}{m\omega\,|\mathbf{r} - \mathbf{r}'|^2} - \frac{1}{\omega^2 s^2}. \tag{10}$$

The first term of an asymptotic expansion in inverse powers of $\frac{m\omega^2 |\mathbf{r} - \mathbf{r}'|^2}{2KT}$ is (ERDELYI, 1956):

$$\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') \approx \frac{n_0 e^2 m^{3/2}(\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}')}{KT(2\pi KT)^{3/2}} \times \left[\frac{-4\pi KT}{m\omega^2 |\mathbf{r} - \mathbf{r}'|^2 h''(s_0)}\right]^{1/2} \frac{\exp\left[\frac{m\omega^2 |\mathbf{r} - \mathbf{r}'|^2}{2KT} h(s_0)\right]}{s_0^5} \tag{11}$$

where s_0 is the value of s for which $h'(s) = 0$:

$$h'(s) = \frac{2KT(i - \nu/\omega)}{m\,|\mathbf{r} - \mathbf{r}'|^2\omega} + \frac{2}{\omega^2 s^3}, \tag{12}$$

$$s_0 = \sqrt[3]{\frac{-m\,|\mathbf{r} - \mathbf{r}'|^2}{\omega KT(i - \nu/\omega)}}, \tag{13}$$

$$h''(s) = \frac{-6}{\omega^2 s^4}. \tag{14}$$

Substituting equations (12) through (14) into equation (11), one obtains

$$\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') \approx \frac{-n_0 e^2 \omega(\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}')}{2\pi(KT)\,|\mathbf{r} - \mathbf{r}'|^3\sqrt{3}}\left(i - \frac{\nu}{\omega}\right) \times \exp\left[\frac{-3m^{1/3}\omega^{2/3}\,|\mathbf{r} - \mathbf{r}'|^{2/3}}{2(KT)^{1/3}}\right]\left(i - \frac{\nu}{\omega}\right)^{2/3} \tag{15}$$

with $\arg(i - \nu/\omega)^{1/3} = 1/3 \arg(i - \nu/\omega)$.

It can be seen from equation (15) that $\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}')$ is approximately antihermitian. It has a small hermitian part proportional to the collision frequency ν. Thus, power transfer between the homogeneous warm plasma and the electromagnetic field is proportional to ν. The same statement has been shown to be true for homogeneous warm plasmas in static magnetic fields except for certain narrow frequency bands at approximately integral multiples of the electron cyclotron frequency. There the power transfer density is inversely proportional to the total effective collision frequency (which includes a Landau type diffusion damping frequency) and the width of the bands is proportional to this total effective collision frequency. Thus the total power density is approximately independent of the collision frequency and represents a direct exchange between the kinetic energy of the plasma electrons and the electromagnetic radiation field.

A1.6 *Conduction kernel for a sheath*

As shown by TONKS and LANGMUIR (1929), ion oscillations will occur in plasmas at frequencies very much smaller than the cut-off above which radio waves can propagate. This frequency can also be made much lower than the relaxation time for the plasma electrons to come to equilibrium among themselves (SPITZER, 1956). Thus ion oscillations form an important class of quasi-static inhomogeneities in plasmas. For this class, the quasi-static magnetic field is zero, the conduction currents just cancelling the displacement currents. The equilibrium among electrons is represented by the Maxwell-Boltzmann distribution:

$$f_0 = n_0\left(\frac{m}{2\pi KT}\right)^{3/2} \exp\left[-\frac{\frac{1}{2}mv^2 - |e|\Phi(\boldsymbol{\tau})}{KT}\right] \quad (1)$$

which satisfies the steady-state Boltzmann equation (A1.1–2) with $\mathbf{H}_0 = 0$, and $\mathbf{E}_0 = -\nabla\Phi$.

For ion oscillations for which the amplitude of the ion displacements is very small compared to the wavelength, the potential satisfies

$$\frac{\partial^2}{\partial x^2}(\ddot{\Phi} + \Omega_p{}^2\Phi) - \frac{1}{\lambda_D{}^2}\ddot{\Phi} = 0 \quad (2)$$

where

$$\Omega_p{}^2 = \frac{4\pi n_0 e^2}{M}, \qquad \lambda_D = \frac{4\pi n_0 e^2}{KT}.$$

The linearly time dependent solution is given by

$$\Phi = -E_0 x \quad (3)$$

where E_0 is independent of x.

The characteristic equations become

$$\tfrac{1}{2}mv_x{}^2 + |e|\,E_0 x = \tfrac{1}{2}mv_x{}'^2 + |e|\,E_0 x' \quad (4)$$

or

$$v_x = -\frac{|e|\,E_0}{2m}s + \frac{x - x'}{s}, \quad (5)$$

and

$$v_y = \frac{y - y'}{s}, \quad (6)$$

and

$$v_z = \frac{z - z'}{s}. \quad (7)$$

From equations (5) through (7), equation (A1.1–16) becomes again

$$\mathscr{J} = \frac{-1}{s^3} \quad (8)$$

Equation (1) yields

$$\nabla_{\mathbf{v}} f_0 = \frac{-n_0 m}{KT}\left(\frac{m}{2\pi KT}\right)^{3/2}\left(\frac{\mathbf{r} - \mathbf{r}'}{s} + \frac{|e|\,\mathbf{E}_0 s}{2m}\right)$$

$$\times \exp\left[\frac{-m}{2KT}\left(\frac{\mathbf{r} - \mathbf{r}'}{s} + \frac{|e|\,\mathbf{E}_0 s}{2m}\right)^2 - \frac{|e|\,E_0 x'}{KT}\right]. \quad (9)$$

Thus the transform kernel is

$$\mathscr{H}(\mathbf{r}, \mathbf{r}', s) = \frac{-n_0 m}{KTs^5}\left(\frac{m}{2\pi KT}\right)^{3/2}\Big[(\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}')$$

$$- \frac{|e|\,\mathbf{E}_0 s^2}{2m}(\mathbf{r} - \mathbf{r}') + \frac{|e|\,s^2}{2m}(\mathbf{r} - \mathbf{r}')\mathbf{E}_0 - \frac{|e|^2 s^4}{4m^2}\mathbf{E}_0\mathbf{E}_0\Big]$$

$$\times \exp\left[\frac{-m}{2KT}\left(\frac{\mathbf{r} - \mathbf{r}'}{s} + \frac{|e|\,\mathbf{E}_0 s}{2m}\right)^2 - \frac{|e|\,E_0 x'}{KT}\right] \quad (10)$$

The conduction kernel is thus given by

$$\mathbf{K}_\omega(\mathbf{r}, \mathbf{r}') = \frac{n_0 e^2}{KT}\left(\frac{m}{2\pi KT}\right)^{3/2}\Big((\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}')$$

$$\times \int_0^\infty \exp\left[\frac{m\omega^2}{2KT}(\mathbf{r} - \mathbf{r}')^2 h_1(s)\right]\frac{ds}{s^5}$$

$$+ \left[-\frac{|e|\,\mathbf{E}_0}{2m}(\mathbf{r} - \mathbf{r}') + \frac{|e|}{2m}(\mathbf{r} - \mathbf{r}')\mathbf{E}_0\right]$$

$$\times \int_0^\infty \exp\left[\frac{m\omega^2}{2KT}|\mathbf{r} - \mathbf{r}'|^2 h(s_1)\right]\frac{ds}{s^3}$$

$$- \frac{e^2}{4m^2}\mathbf{E}_0\mathbf{E}_0 \int_0^\infty \exp\left[\frac{m\omega^2}{2KT}|\mathbf{r} - \mathbf{r}'|^2 h_1(s)\right]\frac{ds}{s}\Big)$$

where

$$h_1(s) = \frac{2KT(i-\nu/\omega)s}{m\omega\,|\mathbf{r}-\mathbf{r}'|^2} - \frac{1}{\omega^2 s^2} - \frac{(\mathbf{r}+\mathbf{r}')\cdot\mathbf{E}_0\,|e|}{\omega^2 m\,|\mathbf{r}-\mathbf{r}'|^2} - \frac{e^2\mathbf{E}_0{}^2 s^2}{4m^2\omega^2\,|\mathbf{r}-\mathbf{r}'|^2}$$

$$h_1'(s) = \frac{2KT(i-\nu/\omega)}{m\omega\,|\mathbf{r}-\mathbf{r}'|^2} + \frac{2}{\omega^2 s^3} - \frac{e^2 E_0{}^2 s}{2m^2\omega^2\,|\mathbf{r}-\mathbf{r}'|^2}$$

$$h_1''(s) = \frac{-6}{\omega^2 s^4} - \frac{e^2\mathbf{E}_0{}^2}{2m^2\omega^2\,|\mathbf{r}-\mathbf{r}'|^2}$$

$$s_0 \approx \sqrt[3]{\frac{-m\,|\mathbf{r}-\mathbf{r}'|^2}{\omega KT(i-\nu/\omega)}} \qquad \text{to order } \mathbf{E}_0{}^2.$$

The first term in the asymptotic expansion is thus

$$\mathbf{K}_\omega(\mathbf{r},\mathbf{r}') \approx \frac{n_0 e^2}{KT}\left(\frac{m}{2\pi KT}\right)^{3/2}\left[\frac{-4\pi KT}{m\omega^2\,|\mathbf{r}-\mathbf{r}'|^2\,h_1''(s_0)}\right]^{1/2} \times \exp\left[\frac{m\omega^2\,|\mathbf{r}-\mathbf{r}'|^2}{2KT}\,h(s_0)\right] \times \left(\frac{(\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{s_0{}^5} + \left[\frac{-|e|\,\mathbf{E}_0(\mathbf{r}-\mathbf{r}')}{2m} + \frac{|e|\,(\mathbf{r}-\mathbf{r}')\mathbf{E}_0}{2m}\right]\frac{1}{s_0{}^3}\right)$$

$$\approx \frac{-n_0 e^2(\omega+i\nu)i}{2\pi KT\sqrt{3}}\exp -\frac{3}{2}\left[\frac{m\omega^2\,|\mathbf{r}-\mathbf{r}'|^2}{KT}\right]^{1/3} \times \left(i-\frac{\nu}{\omega}\right)^{2/3}\exp\left[\frac{(\mathbf{r}+\mathbf{r}')\,.\,\mathbf{E}_0\,|e|}{2KT}\right] \times \left((\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}') + \frac{|e|}{2m}[(\mathbf{r}-\mathbf{r}')\mathbf{E}_0 - \mathbf{E}_0(\mathbf{r}-\mathbf{r}')]\left[\frac{-m\,|\mathbf{r}-\mathbf{r}'|^2}{\omega KT(i-\nu/\omega)}\right]^{2/3}\right)$$

to order $\mathbf{E}_0{}^2$.

A2. THE REGENERATIVE KERNEL

From Maxwell's equations, one obtains

$$\nabla^2\mathbf{E}_\omega(\mathbf{r}) + \frac{\omega^2}{c^2}\,\mathbf{E}_\omega(\mathbf{r}) = \frac{-4\pi i\omega}{c^2}\,\mathbf{J}_\omega(\mathbf{r}) + 4\pi\nabla\rho(\mathbf{r}) \quad (1)$$

The complete solution of this is

$$\mathbf{E}_\omega(\mathbf{r}) = \mathbf{F}_\omega{}'(\mathbf{r}) + \int_\infty \frac{\exp\left[i\frac{\omega}{c}\,|\mathbf{r}-\mathbf{r}'|\right]}{|\mathbf{r}-\mathbf{r}'|} \times \left[\frac{i\omega}{c^2}\,\mathbf{J}_\omega(\mathbf{r}') - \nabla_{\mathbf{r}'}\rho(\mathbf{r}')\right] d^3\mathbf{r}' \quad (2)$$

where

$$\mathbf{F}_\omega{}'(\mathbf{r}) = \oint \frac{\exp\left[i\frac{\omega}{c}\,|\mathbf{r}-\mathbf{r}'|\right]}{|\mathbf{r}-\mathbf{r}'|} \times \left[\frac{i\omega}{c^2}\,J_\omega(\mathbf{r}') - \nabla_{\mathbf{r}'}\rho(\mathbf{r}')\right] d^2\mathbf{r}' \quad (3)$$

$$= \oint \frac{\exp\left[i\frac{\omega}{c}\,|\mathbf{r}-\mathbf{r}'|\right]}{|\mathbf{r}-\mathbf{r}'|}\left[\frac{\partial\mathbf{E}_\omega(\mathbf{r}')}{\partial n} - \mathbf{E}_\omega(\mathbf{r}')\left(\frac{i\omega}{c} - \frac{1}{|\mathbf{r}'-\mathbf{r}|}\,\frac{\partial\,|\mathbf{r}'-\mathbf{r}|}{\partial n}\right)\right] d^2\mathbf{r}' \quad (4)$$

is the field strength due to the surface current and charge densities $J_\omega(\mathbf{r}')$ and $\rho(\mathbf{r}')$ given on the remote boundary. Into equation (2) may be substituted the functional for $\mathbf{J}$ and ρ in terms of $\mathbf{E}$. This gives

$$\mathbf{E}_\omega(\mathbf{r}) = \mathbf{F}_\omega(\mathbf{r}) + \int_\infty K_\omega(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}_\omega(\mathbf{r}')\,d^3\mathbf{r}' \quad (5)$$

where

$$K_\omega(\mathbf{r},\mathbf{r}') \equiv \frac{i\omega}{c^2}\int_\infty \frac{\exp\left[i\frac{\omega}{c}\,|\mathbf{r}-\mathbf{r}''|\right]}{|\mathbf{r}-\mathbf{r}''|}\,\mathbf{K}_\omega(\mathbf{r}'',\mathbf{r}')\,d^3\mathbf{r}'' - \frac{i\omega}{c}\int_\infty \frac{\exp\left[i\frac{\omega}{c}\,|\mathbf{r}-\mathbf{r}''|\right]}{|\mathbf{r}-\mathbf{r}''|^2} \times (\mathbf{r}-\mathbf{r}'')\mathbf{k}_\omega(\mathbf{r}'',\mathbf{r}')\,d^3\mathbf{r}'' \quad (6)$$

$$+ \int_\infty \frac{\exp\left[i\frac{\omega}{c}\,|\mathbf{r}-\mathbf{r}''|\right]}{|\mathbf{r}-\mathbf{r}''|^3} \times (\mathbf{r}-\mathbf{r}'')\mathbf{k}_\omega(\mathbf{r}'',\mathbf{r}')\,d^3\mathbf{r}''$$

with $\mathbf{K}_\omega$ given by equation (A1.1–26) and $\mathbf{k}_\omega$ by equation (A1.1–29) and

$$\mathbf{F}_\omega(\mathbf{r}) = \mathbf{F}'(\mathbf{r}) - \oint \frac{\exp\left[i\frac{\omega}{c}\,|\mathbf{r}-\mathbf{r}'|\right]}{|\mathbf{r}-\mathbf{r}'|}\,1_n\rho(\mathbf{r}')\,d^2\mathbf{r}'$$

REFERENCES

ALLIS W. P. (1956) *Handbuch der Physik* **XXI**, 383–444, Springer-Verlag, Berlin.

APPLETON E. and BARNETT M. (1925) *Electrician* **94**, 398.

ASH E. A. and GABOR D. (1955) *Proc. Roy. Soc.* **A228**, 447.

BROWN S. C. (1958) *Handbuch der Physik* **XXII**, 531–574. Springer-Verlag, Berlin.

DRUMMOND J. E. and WILCOX L. W. (1955) *Seventh Annual Gaseous Electronics Conference;* DRUMMOND J. E. (1956) Ph.D. Thesis, Stanford University, *Resonant Power Transfer*

Within a Plasma in a Magnetic Field, pp. 34–36, University Microfilms document 17714, Ann Arbor, Michigan; (1958) *Phys. Rev.* **110,** 293 and **112,** 1460; SITENKO A. G. and STEPANOV K. M. (1956) *J. Exp. Theor. Phys. U.S.S.R.* **31,** 642 and (1957) *Sov. Phys.* **4,** 512; PRADHAM I. (1957) *Phys. Rev.* **107,** 1222; BERNSTEIN I. G. (1958) *Phys. Rev.* **109,** 10.

DUNLAP O. E. (1944) *Radio's One Hundred Men of Science*, p. 34, Harper, New York.

ERDELYI A. (1956) *Asymptotic Expansions*, pp. 39–41. Dover Publications, New York.

GERWIN R., DRUMMOND J. E. and SPRINGER B. (1960) *Vector Integral Equations*, Boeing Scientific Research Laboratories Report D1-820065.

KELLS L. M. (1947) *Elementary Differential Equations*. pp. 223–227. McGraw-Hill, New York.

LANDAU L. (1946) *J. Phys. U.S.S.R.* **10,** 25.

MOWER L. (1959) *Phys. Rev.* **116,** 16.

REUTER G. E. H. and SONDHEIMER E. H. (1948) *Proc. Roy. Soc.* **A195,** pp. 336.

SPITZER L. (1956) *Physics of Fully Ionized Gases*, Interscience, New York, London.

TONKS L. and LANGMUIR I. (1929) *Phys. Rev.* **33.**

TRICOMI F. G. (1957) *Integral Equations*, Interscience, New York, London, p. 145.

WATSON G. N. (1952) *Theory of Bessel Functions*, pp. 450–453. Cambridge University Press.

WHARTON C. B. (1960) *Plasma Physics*, Chap. 11 (Drummond J. E. Ed.) McGraw-Hill, New York.

VOL
2
196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 109 to 111. Pergamon Press Ltd. Printed in Northern Ireland

EXPERIMENTS ON INTERACTION OF PLASMAS AND MICROWAVES

J. G. LINHART*
CERN, Geneva, Switzerland

Abstract—A short discussion is presented of some experimental work in the field of plasma physics in progress at CERN.

THE theory of propagation of electromagnetic waves on plasma cylinders has been worked out originally by SCHUMANN (1948). In recent years an interest in such plasma waveguides has been aroused owing to their possible use as slow-wave structures in linear accelerators and travelling-wave tubes. The relevant theory has been refined and extended by FAINBERG and his collaborators (FAINBERG, 1956) who have taken into account the effect of magnetostatic fields on the propagation of cylindrical wave.

We have felt that an experiment should be set up representing as nearly as possible the plasma structures treated by the theoreticians.

We have, therefore, designed an apparatus in which:

(a) a cylindrical discharge was well separated from any material walls,

(b) the radial distribution of plasma density resembled a step function, most of the plasma being concentrated within a cylinder whose radius could be varied from 1/2 cm to about 1·2 cm,

(c) the linear plasma density was constant over the length of the discharge.

These requirements were satisfied by a reflex discharge in argon located on the axis of a cylindrical cavity (Figs. 1 and 2). An adjustable magnetic loop coupled to the plasma cylinder output of a local microwave oscillator. Another loop was located at the opposite side of the cavity and near the plasma surface. Two microwave probes could be moved independently in the axial and radial direction (Fig. 3). These probes were used to investigate the pattern of the microwave field inside the cavity.

* Present address: Euratom–CNEN, Frascasi (Roma).

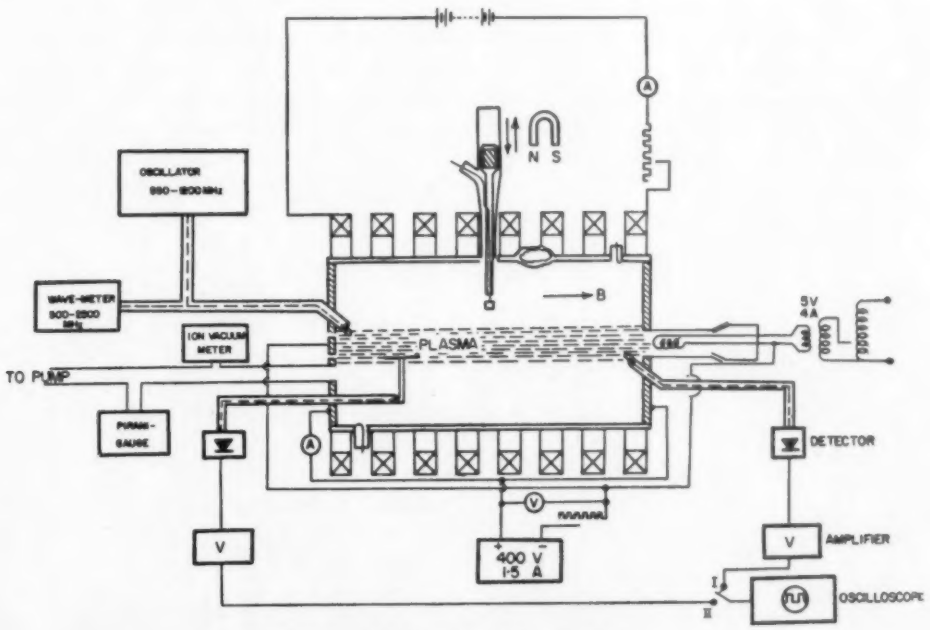


FIG. 1.—Schematic view of the experiment.

EXPERIMENTAL RESULTS ON PLASMA WAVEGUIDES

The first modes detected were the ordinary resonant modes of the copper cavity detuned somewhat owing to the permittivity of the discharge column. From the amount of the detuning we were able to calculate the linear density of electrons in our reflex discharge. The wavelength of the longest cavity mode was about 19 cm, whereas according to theory the plasma modes were expected at longer wavelengths than this.

The second type of mode detected was a coaxial mode, the plasma cylinder playing the role of the inner conductor of a coaxial cable. Indeed, by substituting a copper rod whose diameter was equivalent to that of our plasma column the same mode, having the same pattern and nearly the same frequency, was found.

Lastly, several modes were detected whose field was found to be concentrated near the surface of the plasma cylinder and whose general behaviour indicated that these were the pure plasma modes. Using a movable dipole probe the axial variation in electric fields of these modes was measured. This has shown that the modes within the frequency range of our oscillator were the E_{012}, E_{013} and E_{014} modes (Fig. 4). Comparing the experimental data with those obtained from the theoretically derived dispersion relationships a surprisingly good agreement was found.

Our further objective was to measure the losses in our plasma resonator. So far, we have only found an upper limit to these losses which follows directly from the measured Q-factor of our resonators which was for all three modes in the region between 200 and 300. This Q is, however, composed of two contributions, one being the load Q and the other being the Q of an unconnected plasma resonator. It is only the latter from which the loss of r.f. in plasma can be calculated. These two Q's have not been separated as yet. Such

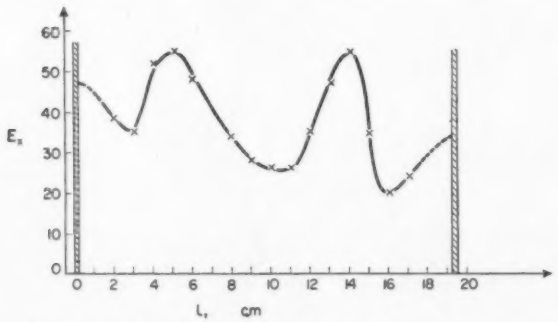


FIG. 4.—Axial distribution of the electric field in the E_{013} mode.
$f_{res} = 732$ MH$_z$ $\lambda_0 = 40{\cdot}8$ cm $\lambda_z = 13$ cm
$p = 4 \times 10^{-4}$ mm Hg
$H = 180$ G
$I_D = 80$ mA.

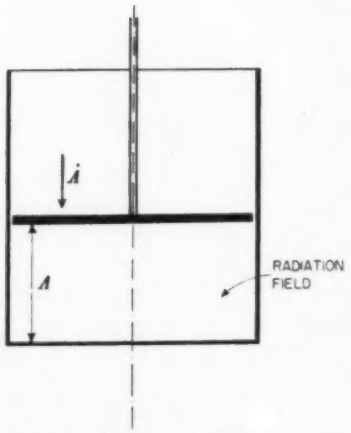


FIG. 5.—Schematic view of a piston-tunable cavity resonator.

a separation will also have to eliminate the effect of small-scale plasma instabilities which cause the resonant frequency of the plasma resonator to shiver. This effect would mask the resonance characteristic of a resonator. Considerations of effects of such frequency-shiver were partly responsible for leading us to investigate a phenomenon of radiation compression, the basic mechanism of which can be described as follows.

Let us consider a piston-tunable cavity-resonator in which a resonant mode of oscillation is excited (Fig. 5). By pushing in the piston the resonant frequency of the mode is increased. If at the same time there are no energy losses in the walls of the cavity one may use the adiabatic theorem for oscillators which states that

$$W/f = \text{constant} \tag{1}$$

where W is the stored energy of the oscillating mode and f is the resonant frequency. Thus, in the ideal adiabatic case the stored energy increases proportionally to the resonant frequency.

In real cavity resonators some energy loss $\dot{W}$ always exists and it is usually described in terms of the quality factor Q,

$$Q = \frac{\omega W}{\dot{W}} . \tag{2}$$

Let us now compare the energy gain, due to the compression of the mode of oscillation, with the losses in the resonator. For a uniform decrease in all dimensions of a resonator we have

$$f/f_0 = \Lambda_0/\Lambda \tag{3}$$

where Λ is some characteristic linear dimension. From equations (1) and (3) we get for the differential of energy gain

$$dW_1 = W\frac{d\Lambda}{\Lambda} \tag{4}$$

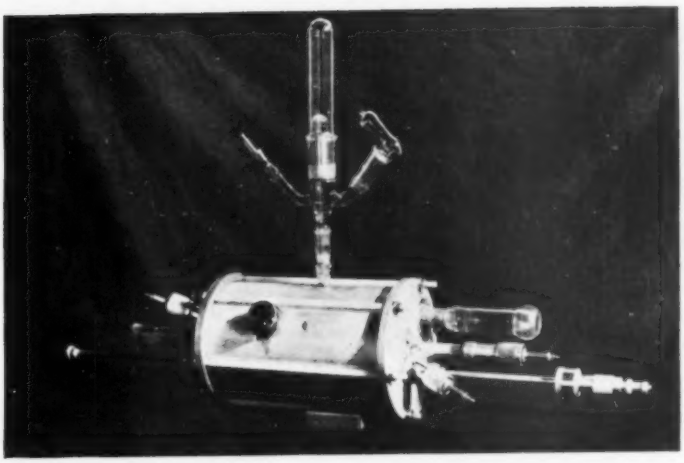


FIG. 2.—Cylindrical cavity used in the experiment.

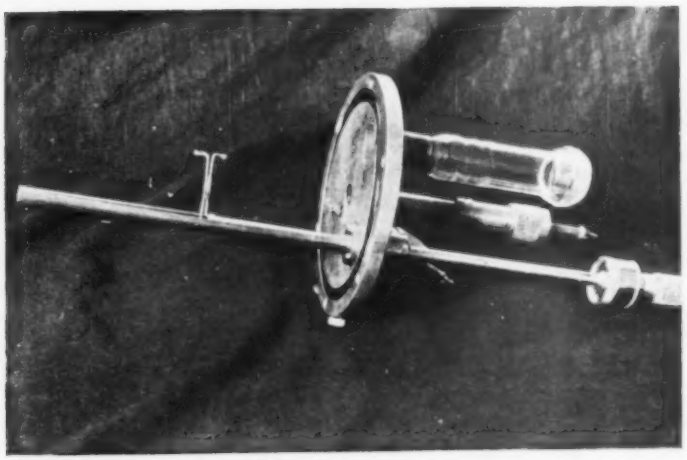


FIG. 3.—View of cavity end plate showing microwave probes.

VOL
2
196

whereas from equation (2) the differential of energy loss is

$$dW_2 = \frac{W\omega}{Q}\,dt. \tag{5}$$

Consequently, the net rate of change of stored energy is

$$\dot{W}_s = W_s\left(\frac{\dot{\Lambda}}{\Lambda} - \frac{\omega}{Q}\right). \tag{6}$$

Thus, r.f. energy generation by radiation compression becomes possible only if the compression speed

$$\dot{\Lambda} > \Lambda\frac{\omega}{Q}$$

or

$$\dot{\Lambda} > \frac{2\pi}{Q}\,c\,\frac{\Lambda}{\lambda} \tag{7}$$

where λ is the free-space wavelength of the oscillating field.

For a resonator oscillating in a principal mode

$$\frac{\Lambda}{\lambda} \sim \frac{1}{2}\,(\dot{\varepsilon}\mu)^{-1/2}$$

and one gets

$$\dot{\Lambda} > \frac{\pi}{Q}\,(\varepsilon\mu)^{-1/2}\cdot c. \tag{8}$$

For an orthodox cavity resonator $Q \sim 10^4$ and the critical speed for radiation compression is therefore about 10^7 cm/sec.

It is clear that the compression speeds required for efficient generation of high radiation pressures cannot be achieved by ordinary tuning methods used in microwave engineering; however, plasma can be driven to these speeds and at the same time can be as good an electrical reflector of microwaves as copper.

In order to investigate the effect of radiation compression experimentally we have designed a cylindrical cavity which could be excited in a certain mode of oscillation and into which a volume of plasma could be shot, incident on the field of the resonant mode. Preliminary experimental results have shown the existence of the phenomenon of radiation compression.

Acknowledgements—I would like to thank Messrs. W. Zych and L. Th. Ornstein for their enthusiastic help with the experiments.

REFERENCES

Fainberg Ya. I. (1956) *CERN Symposium*, vol. 1, p. 84.
Schumann W. O. (1948) *Ber. Bayer. Akad. Wiss.*, 255.

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 112 to 116. Pergamon Press Ltd. Printed in Northern Ireland

CONFINEMENT OF AN ELECTRON BEAM BY OSCILLATING ELECTROMAGNETIC FIELDS*

E. S. WEIBEL and G. L. CLARK
Physical Research Laboratory, Space Technology Laboratories Inc.,
Los Angeles, California

Abstract—It has been shown theoretically that a charged particle can be bound in stable orbits about the axis of a circular wave guide by the fields of the TE_{01} mode at cut-off (WEIBEL, 1959). This result was verified experimentally. The apparatus consists of a 20-cm long section of a circular wave guide—properly terminated—into which is fed radio-frequency power from a magnetron at $f = 9{\cdot}29$ kMc/s in 2 μsec pulses. A uniform TE_{01} field is excited over the entire 20-cm length of the guide. An electron beam which is injected axially at one end of the cavity diverges strongly due to coulomb repulsion if no radio-frequency power is applied. During the radio-frequency pulse, however, the beam is confined to the axis and 90 per cent of the current is collected by an axial probe of 0·3-cm diameter which is located at the other end of the guide. Between pulses when no radio frequency is applied, no measurable current reaches the probe. In its trajectory from the gun to the probe each electron spends about 170 radio-frequency periods in the focusing field.

1. INTRODUCTION

In recent years the confinement of a plasma by radio-frequency fields has been discussed by several authors (KNOX, 1957; WEIBEL, 1958; BOOT, SELF and SHERSBY-HARVIE, 1958; WUERKER, SHELTON and LANGMUIR, 1959; GOOD, 1953; PAUL and STEINWEDEL, 1953; PAUL and RAETHER, 1955; PAUL, REINHART and VON ZAHN, 1958; BOOT and SHERSBY-HARVIE, 1957; WUERKER, GOLDENBERG and LANGMUIR, 1959; MILLER, 1959). Although the original motive for this work may have come from efforts to contain a fusion reaction, it became clear immediately that such confinement would not result in an economic reactor of reasonable size due to the enormous power required to maintain the containing fields.† Even worse, it turned out that radio-frequency confinement has its own instabilities for high-density plasmas (WEIBEL, 1958). However, the confinement of tenuous plasmas by oscillating electromagnetic fields is still of great interest since conditions do exist under which it is stable.

Perhaps the most important feature of confinement by oscillating fields is the possibility of creating effective 'potential wells' for charged particles, that is, potentials which can have true minima. This is not possible for either electrostatic or magneto-static fields.

In a standing electromagnetic wave the expression for the potential is‡

$$\psi(\vec{r}) = \frac{e^2E^2(\vec{r})}{4m\omega^2} \tag{1}$$

where $E(\vec{r})$ is the electric field amplitude and ω the radian frequency while e and m are the charge and the mass of the particle. The nodal points, lines or nodal surfaces of the electric field, thus form the minima of the potential. However, since

$$\vec{F} = -\text{grad } \psi \tag{2}$$

represents only the time-averaged force, stability is not automatically guaranteed. A stability condition of the form

$$\omega^2 \geqslant \alpha \frac{e}{m} \left|\frac{\partial E_i}{\partial X_k}\right|_{\max} \tag{3}$$

must be imposed where α is a numerical factor close to unity, depending on the geometry of the field.

The significance of this condition is simply to require the oscillations of the field to be more rapid than the resulting oscillation of the particle trapped in the well.

Not only single particles but a plasma can be stably confined to a small volume about the nodes of the electric field, provided that the number of particles in the well is so small that it does not appreciably affect the field:

$$\frac{e^2n_e}{m_e\omega^2} + \frac{e^2n_i}{m_i\omega^2} < 1\,. \tag{4}$$

Plasma confinement of this type has been considered by several Russian authors (GAPANOV and MILLER, 1958a; GAPANOV and MILLER, 1958b; ASKARYAN, 1958) as a means of focusing and accelerating neutral plasmas to high energies. Space charge would thus no longer limit the attainable beam currents as in conventional high energy particle accelerators.

It seemed desirable to test the focusing property of an oscillating field in a simple experiment. For ease of

* Prepared for the Air Force Ballistic Missile Division Headquarters Air Research and Development Command under Contract AF 04(647)–309.

† No such general statement can be made for containment methods which employ combinations of static and oscillatory fields.

‡ Rationalized units, velocity of light $c = 1$.

instrumentation this experiment which will be described in Section 3 was performed on an electron beam rather than on a neutral plasma. The idea is not to compete with far simpler methods of electron beam focusing, but to test the confining property of the field.

2. THEORY

Of particular interest is a cylindrical geometry since it can be analysed in detail and because at the same time it lends itself to an experiment which clearly demonstrates the stability of the confinement.

The fields of the mode of a circular wave guide at cut-off

$$E_\phi = -B_0 J_1(\omega r)\cos(\omega t), \tag{5}$$

$$B_z = B_0 J_0(\omega r)\sin(\omega t), \tag{6}$$

provide according to (1) a potential well,

$$\psi(r) = \frac{e^2 B^2{}_0}{4m\omega^2} J_1{}^2(\omega r) \tag{7}$$

with a radius of

$$r_m = 2{\cdot}4/\omega. \tag{8}$$

At this radius ψ assumes its maximum value

$$\psi_{\max} = \psi(r_m) = 0{\cdot}084m\left(\frac{eB_0}{m\omega}\right)^2. \tag{9}$$

The single particle motion has been analysed (WEIBEL, 1959) and was found to be stable if

$$\omega > 0{\cdot}433\,\frac{eB_0}{m}. \tag{10}$$

Since an electron beam was used in the experiment, it is necessary to take into account the effects of space charge. This space charge gives rise to a repulsive electrostatic potential ϕ which satisfies Poisson's equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = en(r), \tag{11}$$

where $n(r)$ is the density of electrons. The electrostatic repulsion, the confining force, and the pressure gradient must be in equilibrium. Using for the pressure $p = nkT$ one obtains

$$-enE_r - n\frac{d\psi}{dr} - kT\frac{dn}{dr} = 0$$

where the radial electric field is given by

$$E_r = -\frac{d\phi}{dr} = -\frac{e}{r}\int_0^r n(r')r'\,dr'. \tag{12}$$

Equation (12) can be divided by n and then integrated to give

$$n(r) = n_0 \exp\left[-\frac{1}{kT}(\psi - e\phi)\right]. \tag{13}$$

This distribution represents equilibrium only in an approximate sense, since forced oscillations of the confining field are superimposed on the average motion of the particles in the potential $\psi - e\phi$. Thus the temperature T merely represents the kinetic energy of the electrons, but does not imply thermal equilibrium.

In two limiting cases the two equations (11) and (13) can be solved explicitly. If T approaches zero

$$T \to 0 \tag{14}$$

then

$$\phi \to \frac{1}{e}\psi \tag{15}$$

and

$$n = \frac{1}{e^2}\frac{1}{r}\frac{d}{dr}\left(\frac{d\psi}{dr}\right) \quad r < a \tag{16}$$

$$n = 0 \qquad r > a \tag{17}$$

where a is an arbitrary radius not exceeding the radius of the well, $a < r_m$. In this case the electron beam has a sharp boundary and the electrostatic repulsion is balanced everywhere exactly by the confining force. For any positive temperature the density n must be less than that given by (16). In the limit of very low density, that is, high temperature, the coulomb repulsion becomes unimportant and one obtains

$$n(r) = n_0 \exp -\frac{\psi}{kT}. \tag{18}$$

In the general case the two equations (11) and (13) can easily be integrated numerically (WEIBEL, 1958b); however, it is more useful for the present purpose to obtain an approximate analytical expression for the radius of the electron beam as a function of its density, of the temperature, and of the applied field. Thus, the following approximations will be made:

$$\psi = \frac{1}{2}m\omega_0{}^2 r^2, \quad \omega_0 = \frac{eB_0}{\sqrt{8}m}, \quad r \ll 1/\omega. \tag{19}$$

Integrating (12) from zero to infinity one obtains

$$e^2\int_0^\infty \frac{dr}{r}n(r)\int_0^r n(r')r'\,dr' - m\omega_0{}^2\int_0^\infty n(r)\,r\,dr + [n(0) - n(\infty)]kT = 0 \tag{20}$$

The radius a of the beam shall now be defined by

$$\pi a^2 n_0 = N = \int_0^\infty n(r)2\pi r\,dr \tag{21}$$

where N is the number of electrons per unit length of

the beam. The first integral of (20) is clearly of the form

$$\frac{\beta}{4} e^2 n_0^2 a^2 \tag{22}$$

where β is a number depending on the form of $n(r)$. In the two limiting cases discussed above β can be calculated. In the low temperature limit $n(r)$ is according to (16) and (19) a step function

$$n = n_0 = 2\frac{m\omega_0^2}{e^2} \quad r < a \tag{23}$$

$$n = 0 \qquad r > a \tag{24}$$

so that $\beta = 1$. In the high temperature limit one obtains from (18) and (19)

$$n = n_0 \exp - \frac{m\omega_0^2 r^2}{2kT} \tag{25}$$

for which β turns out to be log 2. Hence

$$\log 2 < \beta < 1 \tag{26}$$

and equation (20) becomes

$$(\omega_0 a)^2 \left(\frac{\beta}{4}\frac{e^2 n_0}{m\omega_0^2} - \frac{1}{2}\right) + \frac{kT}{m} = 0. \tag{27}$$

It follows immediately that

$$n \leqslant n_0 \leqslant \frac{2}{\log 2} \cdot \frac{m\omega_0^2}{e^2}. \tag{28}$$

In terms of N, the number of electrons per unit length of the beam one obtains for its radius, a:

$$a^2 = \frac{1}{\omega_0^2}\left[2\frac{kT}{m} + \frac{\beta}{2\pi}\frac{e^2 N}{m}\right] \tag{29}$$

Since the space charge has modified the force acting on each electron, it is necessary to re-examine the stability of the individual orbits. The same technique which was used to determine the stability of single particle orbits (WEIBEL, 1959) leads again to a Mathieu equation

$$\frac{d^2}{dt^2}\begin{Bmatrix} x \\ y \end{Bmatrix} + \left[\omega_0^2 - \frac{\omega_p^2}{2} - \omega_0^2 \cos(2\omega t)\right]\begin{Bmatrix} x \\ y \end{Bmatrix} = 0 \tag{30}$$

where

$$\omega_0 = \frac{eB_0}{\sqrt{8}m}, \quad \omega_p = \sqrt{\frac{e^2 n}{m}} \tag{31}$$

where x and y are the Cartesian co-ordinates of the particle in a system which executes an angular oscillation

$$\phi = \frac{eB_0}{m\omega} \cos \omega t \tag{32}$$

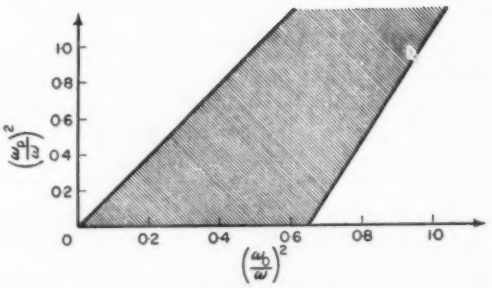


FIG. 1.—The stability diagram for equation (30). The shaded portion represents the region of stability.

about the z-axis. The stability diagram of Fig. 1 was constructed from a table of Mathieu functions. There are other stable regions of equation (30) than the one shown, but it appears unlikely that these could be reached experimentally. The stable region, as determined from Mathieu's equation, is unbounded; however, the condition that the field penetrate the plasma implies the additional constraint

$$(\omega_p/\omega)^2 \ll 1. \tag{33}$$

So far it is not known what the exact limit for ω_p/ω is nor whether such a restriction is indeed necessary.

3. THE EXPERIMENT

To test experimentally the predictions of the theory a field of the type (5) (6) has to be produced. This requires some care since, strictly speaking, a mode at cut-off can only exist in an infinite guide. If a circular guide were closed off at $z = \pm A$ by two conducting disks, the resulting field would depend sinusoidally on z and, in addition, a radial magnetic field would appear.

However, substantial uniformity of the field can be achieved if the guide is terminated at both ends by resonators (Fig. 2) which consist of wave guides of a radius slightly larger than that of the main section. Since the TE_{01} mode is above cut-off in these resonators the field varies in the z direction and can be made to match the boundary conditions. Detailed calculations (WEIBEL, 1958c) show that field uniformity of $\pm 3{\cdot}5$ per cent can be achieved by this method.

Since the central section has to support a mode at cut-off over a distance of 6·2 free-space wavelengths even small deviations from the true dimensions can result in an exceedingly non-uniform field. Therefore, the bore of this section was made cylindrical within a tolerance of 10^{-4} : 1.

Microwave power from a tuneable magnetron at 9·29 kMc is fed into one of the end resonators (Fig. 2) through a coupling window. The frequency of the magnetron and the position of the end plates are

VOL.
2
1961

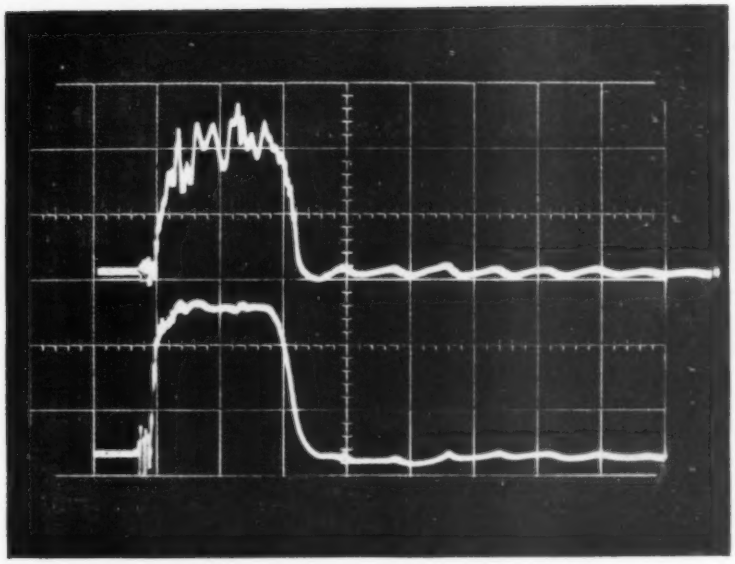


FIG. 3.—Typical oscillogram traces of the r.f. pulse (lower) and the collected current (upper). Sweep speed = 1 μsec per major horizontal division.

VOL
2
196

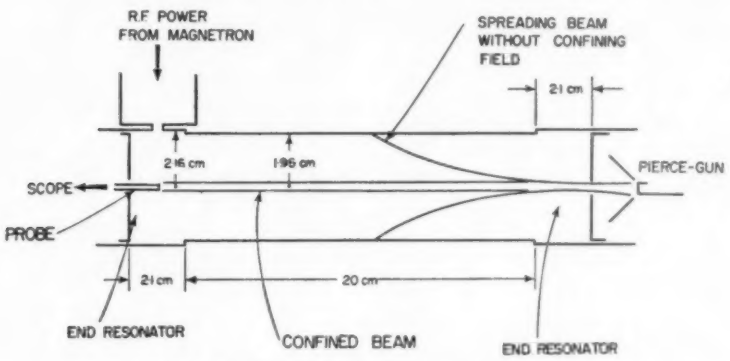


FIG. 2.—Schematic view of experimental apparatus.

adjusted to give the best uniformity of the field. Measurements performed on the finished cavity indicated a variation of the field along the z-axis of ± 5 per cent.

The magnetron is pulsed at a repetition rate of 400 c/s delivering 250 kW of power in 2 μsec pulses. At this power, the magnetic field amplitude on the axis is $B_0 = 300$ gauss, while the electric field reaches a maximum of 52 kV/cm. Thus according to equation (19) $\omega_0 = 1{\cdot}89 \times 10^9$ sec $= (15{\cdot}9$ cm$)^{-1}$ and $\omega_0/\omega = 3{\cdot}35 \times 10^{-2}$. The depth of the potential well (7) becomes $\psi_{\max} = 400$ eV.

An electron beam of 0·3 mA is injected axially into one end resonator at 400 volts. This accelerating voltage is kept low to let the electrons interact with the field for a long time. At this voltage the electrons travel with a velocity of $1{\cdot}1 \times 10^7$ cm/sec through the cavity, thus they remain under the influence of the field for 170 r.f. periods. The density per unit length of the beam becomes $N = 1{\cdot}7 \times 10^6$ cm^{-1}.

When no r.f. field is applied, the beam spreads rapidly due to coulomb repulsion and almost all electrons strike the cavity wall. During the microwave pulses, however, the beam is focused and 90 per cent of the electrons are collected by an axial probe of 3 mm diameter, which is located at the opposite end of the cavity. The collected current is returned to ground through a 1000-ohm resistor, and the voltage developed across this resistor is directly applied to an oscilloscope. The lower trace of Fig. 3 shows the r.f. pulse while the upper one represents the collected current. (The wiggles on the current trace are not yet understood in detail but are presumably due to some oscillations of the beam in the potential well.) The traces of Fig. 3 constitute clear-cut evidence of the focusing by the oscillating field.

Equation (29) will now be verified using the experimental results. With the few measurements made so far this can be done only approximately.

To calculate the beam radius, a, from equation (29) the temperature T should be known. It can be estimated by examining how the beam is injected into the cavity. The anode aperture of the gun has a radius of 1 mm. Since the confined beam has roughly the same radius as the beam emerging from the gun the transverse energy of the electrons must be about the same as at the anode. From the geometry of the particular Pierce gun which was used one obtains for the ratio of the transverse velocity $v_\perp$ to the longitudinal velocity $v_\parallel$ of the electrons the value $v_\perp/v_\parallel = 1/10$. Hence the temperature T corresponding to this transverse motion equals

$$T = \left(\frac{v_\perp}{v_\parallel}\right)^2 \frac{eV}{k}$$

where V is the accelerating voltage of the gun. With $V = 400$ volts, the temperature becomes 4 electron volts. Inserting the values of ω_0, N, and T into equation (29) one first notices that the term involving N is negligible. Thus the internal pressure of the beam is primarily due to the transverse motion of the electrons so that the distribution (25) applied. For the beam radius one calculates now from (29) the value $a = 0{\cdot}63$ mm and the electron density on the axis becomes $n_0 = 1{\cdot}3 \times 10^8$ cm^{-3}. Thus one obtains for the density distribution (25)

$$n(r) = 1{\cdot}3 \times 10^8 \text{ cm}^{-3} \exp\left[-\left(\frac{r}{0{\cdot}63 \text{ mm}}\right)^2\right]. \quad (34)$$

One concludes that nearly all electrons should be collected by the probe of 1·5 mm radius. Actually a considerable fluctuation in the collected current was observed. However, the peaks of the upper trace of Fig. 3 approach full input current. On the average about 90 per cent of the current was collected.

With an electron density of 10^8 cm^{-3} the inequality (33) is well satisfied since $(\omega_p/\omega)^2$ is about 10^{-4}. For this reason no detuning of the cavity was observed.

While this experiment has shown satisfactory agreement with the theory it should be extended in two directions. Firstly one should attempt to suspend a neutral plasma in such a field, secondly, the limits of stability as shown in Fig. 2 should be explored. This will require varying the quantities ω_p/ω and ω_0/ω over wide ranges.

Acknowledgement—The author is indebted to Dr. R. F. WUERKER for many stimulating discussions on radio-frequency plasma containment.

REFERENCES

ASKARYAN G. A. (1958) *Sov. Phys. JETP*, **36,** 430.

BOOT H. A. H. and SHERSBY-HARVIE R. B. R. (1957). *Nature, Lond.* **180,** 1187.

BOOT H. A. H., SELF S. A. and SHERSBY-HARVIE R. B. R. (1958) *J. Electron. Cont.* **4,** 434.

GAPANOV A. V. and MILLER M. A. (1958a) *Sov. Phys. JETP* **34,** 168.

GAPANOV A. V. and MILLER M. A. (1958b) *Sov. Phys. JETP* **34,** 515.

GOOD M. L. (1953) University of California Report UCRL-4146.

KNOX F. B. (1957) *Aust. J. Phys.* **10,** 221, 565.

MILLER, M. A. (1959); *JETP* **36,** 1909; *Sov. Phys. JETP* 1358.

PAUL W. and RAETHER M. (1955). *Z. Phys.* **140,** 262.

PAUL W. and STEINWEDEL H. *Z. Naturf.* **8a.**

PAUL W., REINHARD, H. P. and ZAHN U. VON (1958). *Z. Phys.* **152,** 143.

WEIBEL E. S. (1958a) *J. Electron. Cont.* **5,** 435.

WEIBEL E. S. (1958b) *The Plasma in a Magnetic Field,* (Ed. LANDSHOFF R. K.) pp. 60–76. Stanford University Press.

WEIBEL E. S. (1958c) *A Resonant Cavity with a Uniform Field Section,* Space Technology Laboratories Report, GM-TR-0127-00017.

WEIBEL E. S. (1959) *Phys. Rev.* **114,** 18.

WUERKER R. F., GOLDENBERG H. M. and LANGMUIR R. V. (1959). *J. Appl. Phys.* **30,** 441.

WUERKER R. F., SHELTON H. and LANGMUIR R. V. (1959) *J. Appl. Phys.* **30,** 342.

VOL
2
196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 117 to 118. Pergamon Press Ltd.

PLASMA OSCILLATIONS

K. G. EMELEUS and D. W. MAHAFFEY*
Queen's University, Belfast, N. Ireland

Abstract—This paper will give a survey of some experiments on the kMc/s oscillations, believed mainly longitudinal, generated when electron beams traverse a plasma, touching on the evidence they provide for the mechanism of collapse of directed motion and its bearing on astrophysical and heavy-current problems and the probable importance of electron concentration gradients evidenced from WEHNER's and our experiments.

THE experiments which we have done in this laboratory on the kMc/s oscillations generated when an electron beam passes through a plasma in gas at a pressure of a few microns are in the main developments of the pioneer experiments of DITTMER (1926), TONKS and LANGMUIR (1929), PENNING (1926) and MERRILL and WEBB (1939). That is, they have been made principally on progressive waves and more or less isolated standing wave packets, rather than on the more fully-developed, standing-wave systems produced by WEHNER (1950) and by LOONEY and BROWN (1954). Apart from one attempt by ARMSTRONG and EMELEUS (1949), we have not concerned ourselves with optimum conditions for production of microwave power, as has been done with considerable success by WEHNER.

It is convenient to use a tube with moveable probe which can also act as an oscillation pick-up and to have at least one main electrode, either anode or cathode, moveable. An approximately parallel electron beam, which also maintains the plasma, can be obtained by using a flat oxide-coated cathode. It is not easy to apply normal good microwave techniques to such a system, and this side of the experiments merits further critical examination.

A large amount of information about the oscillations which are due fundamentally to beam-plasma electron interaction, has been obtained. The most recent results obtained in our laboratory have been described by MAHAFFEY (1959). The oscillations in our tubes normally require a minimum length of beam to be present if they are to occur. The minimum length increases as the current decreases, other parameters being kept as far as possible the same. On onset, however, the oscillations occur through most of the cathode-sheath-anode interspace occupied by the beam (the plasma practically fills the tube). On increasing the current, or anode-cathode separation, the amplitude of the oscillations generally increases, and at a certain stage they detach from the cathode. A quiescent layer of plasma then exists for a few mm from the cathode, and oscillations occur principally in a thinner layer bounding the quiescent layer, to its anode side.

It is possible to understand the phenomena in general terms from the theory of excitation of linear longitudinal electrostatic waves, as developed in particular by LAMPERT (1956) and by SUMI (1959). Some details of the wave-patterns are probably due to formation of a local standing-wave component by the probe (BAILEY and EMELEUS, 1955), which may in some instances play a major role in stabilizing the system.† Existing available theory is inadequate for two reasons. First, the oscillations may be of large amplitude and non-linear; it is not uncommon for beam electrons with initial energies of order of 25 eV to be modulated by as much as 10 to 15 eV in transit through the detached oscillating sheet. Second, there is considerable evidence, first apparently adduced by ALLIS (1957) and later checked in our laboratory (MAHAFFEY, 1959; EMELEUS, MAHAFFEY and MCCULLAGH, 1958) that the growth and decay of the oscillations in space is connected with plasma electron concentration *gradients* in an even more complex way than was noted by WEHNER. At the time of writing, although some progress has been made with non-linear theory by SEN (1955), and by STURROCK (1957), and with the theory of inhomogeneous plasmas by BOHM and GROSS (1949), theory is insufficiently developed, and possibly the experiments are not directed exactly along the right lines, for a satisfactory check-up to be made between the two.

The particular point to which it is desired to call

* Present address: Plasma Physics Laboratory, Boeing Airplane Co., Seattle, Washington.

† This possibility has been strongly urged by Dr. E. U. GORDON in private communications to one of us (K. G. E.), which are gratefully acknowledged.

attention here is the part played by these oscillations in the randomization of the energy and direction of motion of initially nearly mono-energetic electron beams as they pass through a plasma. This possibility was envisaged by LANGMUIR, but has not until recently received much further notice. It seems reasonable that, in any electrical friction between beam and plasma which ultimately slows down and scatters the beam, there should be an intermediate oscillatory stage when the beam-loaded plasma (or plasma-loaded beam) is capable of oscillating and transmitting waves.

That this does in fact occur is made highly probable by the experiments mentioned, in which there has been found to be a close association between occurrence of oscillations and the initial stages of randomization of the beam. Moreover, the change in direction of the beam can be demonstrated visually in several ways by the light it excites in the gas over and above the general plasma luminosity (ALLEN, BAILEY and EMELEUS, 1955) and the change in energy of the beam shown not only electrically but again also optically, from the spectra excited (ALLEN, 1955).

The part which may be played by electron oscillations in thermalization of directed motion has been brought out recently by PARKER (1958), in his study of colliding plasmas. This shows also the significant fact that plasma electron oscillations may be followed by and coupled with plasma ion oscillations, of lower frequency. It is not yet clear if this always occurs, i.e. if plasma electron oscillations can exist alone, or if they are always accompanied by ion oscillations. We have noticed a frequent association of the two in our experiments, but have been inclined to look on them as independent or weakly coupled phenomena. We had however realized that the transverse scattering of the beam in a detached oscillating layer (the 'moniscus' ALLEN, BAILEY and EMELEUS, 1955) might possibly arise from transverse ion oscillations there.

Possible astrophysical applications of this work thus include the randomization of galactic energies in intergalactic collisions, and the emission of radio waves in these collisions, which becomes perhaps less difficult to understand if transverse scattering occurs, as in our experiments. There are also obvious tentative applications to the heating of thermonuclear plasmas by degradation in similar ways of the energy of directed electron or ion beams.

REFERENCES

ALLEN T. K. (1955) *Proc. Phys. Soc.* **A68,** 696.

ALLEN T. K., BAILEY R. A. and EMELEUS K. G. (1955) *Brit. J. Appl. Phys.* **6,** 320.

ALLIS W. P. (1957) *U.R.S.I.* (Boulder, Col.). Unpublished.

ARMSTRONG E. B. and EMELEUS K. G. (1949) *Proc. Inst. Elect. Engrs.* **96,** 197.

BAILEY R. A. and EMELEUS K. G. (1955) *Proc. Roy. Irish Acad.* **57,** 53.

BOHM D. and GROSS E. P. (1949) *Phys. Rev.* **75,** 1864.

DITTMER A. F. (1926) *Phys. Rev.* **28,** 507.

EMELEUS K. G., MAHAFFEY D. W. and MCCULLAGH G. (1958) *Phys. Rev.* **112,** 1052.

LAMPERT M. A. (1956) *J. Appl. Phys.* **27,** 5.

LOONEY D. H. and BROWN S. C. (1954) *Phys. Rev.* **93,** 965.

MAHAFFEY D. W. (1959) *J. Electron. Cont.* **6,** 193.

MAHAFFEY D. W. and EMELEUS K. G. (1958) *J. Electron. Cont.* **4,** 301.

MERRILL H. J. and WEBB H. W. (1939) *Phys. Rev.* **55,** 1191.

PARKER E. N. (1958) *Phys. Rev.* **112,** 1429.

PENNING F. M. (1926) *Physica* **6,** 241.

SEN H. K. (1955) *Phys. Rev.* **97,** 847.

STURROCK P. A. (1957) *Proc. Roy. Soc.* **A242,** 277.

SUMI M. (1959) *Phys. Rev. (Letters)* **2,** 37, (and several Japanese papers).

TONKS L. and LANGMUIR I. (1929) *Phys. Rev.* **33,** 195.

WEHNER G. (1950) *J. Appl. Phys.* **21,** 62 (and other refs.).

VOL 2 196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 119 to 134. Pergamon Press Ltd. Printed in Northern Ireland

MAINTENANCE OF EQUILIBRIUM BY INSTABILITIES

O. BUNEMAN
Stanford Electronics Research Laboratories; and Peterhouse, Cambridge

Abstract—Collective interactions can provide a rapid mechanism for restoring grossly non-Maxwellian distributions to near-Maxwellian in that electrodynamic instabilities will build up to high level within a few plasma periods. Computation of the further development of the instabilities in the non-linear regime shows an approach to statistical randomness.

1. COLLECTIVE INTERACTIONS

IN hot plasmas, individual collisions are too rare to produce and maintain equilibrium efficiently. However, distributions other than near-Maxwellian are so very improbable and the Coulomb interaction between charges remains so very important even at great separation that we must 'explore another avenue' for establishing equilibrium. 'Collective' interactions, i.e collisions of charge-in-bulk, would seem to provide a path by which grossly non-Maxwellian distributions might be restored to near-Maxwellian quickly. The time scale for this process would be given by the typical period for collective processes $2\pi/\omega_p$. This is many powers of 10 shorter than the collision interval. Such a rapid mechanism would rescue magnetohydrodynamics from its failure at long mean free paths.

2. SMALL PERTURBATIONS AND LANDAU DAMPING

Small departures from thermal equilibrium have been analysed extensively, in the theory of 'plasma oscillations,' ignoring individual collisions altogether and taking only collective interactions into account. In uniform Maxwellian plasmas, with and without magnetic field, departures certainly do not grow larger: these plasmas are 'stable'. But a rapid return to the strict Maxwellian state does not take place in that the perturbations are not necessarily damped.

Only certain types of perturbations are damped by collective effects, viz. by the LANDAU (1946) mechanism: departures from neutrality whose spatial extent does not greatly exceed the Debye-length will be damped out within times of the order $2\pi/\omega_p$. However, LANDAU's theory only predicts return to neutrality, not to the Maxwellian velocity distribution. Initially neutral departures remain undamped.

There are further exceptions to LANDAU damping. Disturbances which depend upon velocity in a singular manner (but which are, nevertheless, 'infinitesimal' in a sense), remain undamped and non-neutral forever (VAN KAMPEN, 1955). Likewise, non-infinitesimal disturbances can be constructed which persist indefinitely (BOHM and GROSS, 1949; ALLIS, 1958; BERNSTEIN, GREENE and KRUSKAL, 1957). Finally a magnetic field normal to the wave-vector characterizing the spatial pattern of the disturbance will prevent LANDAU damping (BERNSTEIN, 1958).

In general, therefore, we shall still have to invoke individual collisions for eliminating *small* departures from equilibrium. In other words, once a plasma gets close to equilibrium it will take a long time, namely a few collision intervals, to get much closer.

3. LARGE PERTURBATIONS

The problem is: do arbitrary distributions of ions and electrons in phase space (configuration plus velocity space) show any tendency to develop towards the uniform Maxwellian distribution as a result of collective interactions?

The general initial distribution is, of course, unlikely to be in equilibrium even in the restricted sense of being 'static,' i.e. maintaining itself, in the *absence* of collisions. It will change and either approach a static distribution or maintain a cyclic behaviour (with many incommensurable periods, presumably). We suspect that the static distribution and the mean of the cyclic distribution will bear some resemblance to Maxwellian.

In the absence of a general proof of this conjecture we might investigate, first, what simple static and cyclic distributions are possible and then whether these distributions are stable. Again, we suspect that those which differ greatly from Maxwellian are unstable and will therefore not be approached in general.

4. STATIC DISTRIBUTIONS

Even all static distributions form too large a class for easy analysis. Attention is drawn to the infinite variety of one-dimensional distributions described by BERNSTEIN, GREEN and KRUSKAL (1957) (from which

cyclic distributions are obtained by translation with uniform velocity).

Let us restrict our analysis further to static distributions which are uniform in configuration space (and which must therefore be neutral). Only in velocity space we assume substantial departure from Maxwellian.

Isotropic non-Maxwellian distributions (for instance a hollow shell distribution in velocity space) can readily be tested for stability. They are certainly stable in the absence of magnetic fields (see for instance, BUNEMAN, 1958). Whether they are unstable in a uniform magnetic field to perturbations with transverse propagation constant is under investigation in Cambridge. It seems unlikely, and a stability proof may well have been produced elsewhere already.

5. ANISOTROPIC DISTRIBUTIONS

Collective effects, we conclude, will at best be able to eliminate anisotropies. Again, a general analysis is too difficult but some progress can be made with (i) multiple interstreaming (ii) one-dimensional anisotropies.

These two cases are perhaps the most relevant from a practical point of view. Grossly non-Maxwellian distributions arise when plasmas are created by injection (i.e. by producing streams) and when particles 'run away' in the direction of an applied electric field.

Discrete interpenetrating streams, each of them 'cold', lead to instabilities because the dispersion formula,

$$\sum \omega_{pv}^2/(\omega - ku_v)^2 = 1 \quad (v = \text{stream index}) \qquad (1)$$

may have up to $v - 1$ pairs of complex ω-roots. The imaginary parts will be of the order of ω_{pv}, and the distribution will change into something different in times of the order $2\pi/\omega_{pv}$.

Only when the discrete streams become so numerous that they are better described by a continuous family do the complex roots coalesce into the real axis and we get the situation discussed in the theories of LANDAU (1946) and VAN KAMPEN (1955). The sum above becomes an integral, the ω_{pv} become infinitesimal and the build-up times infinitely long.

6. ONE-DIMENSIONAL DISTRIBUTIONS

A good deal is known about the instability of only two-streams. This is the special case

$$f(v) = \tfrac{1}{2}\,\delta(v - u_1) + \tfrac{1}{2}\,\delta(v - u_2)$$

of a one-dimensional velocity spectrum. The analysis is still tractable when the two streams are not cold but are each separately Maxwellian, i.e. when the singular delta-functions are regular Gaussians.

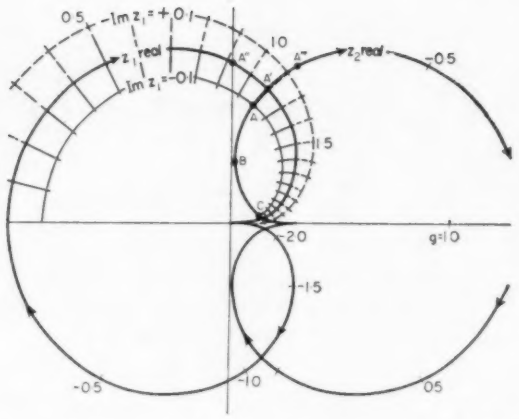


FIG. 1.—Contours of constant real and imaginary parts of w in the complex plane of the integral $\int (v - w)^{-1} f'(v)\,dv$ for Gaussian $f(v)$.
z_1 is proportional to w. The curve 'z_2 real' is associated with a superimposed displaced second Gaussian.

Simple cases of double streaming arise when a strong current is set up and the electrons acquire an appreciable drift relative to the ions, or when two plasmas collide, as in a shock (PARKER, to be published). The dispersion formula then becomes

$$\int (\omega - kv)^{-2} f(v)\,dv = \omega_p^{-2} \qquad (2)$$

where the ions can be included in the overall distribution function $f(v)$ combined from electrons and ions by weighting down the ion density with the factor m/M. Integrating by parts in (2) yields

$$\int (v - w)^{-1} f'(v)\,dv = (k/\omega_p)^2; \quad w = \omega/k \qquad (3)$$

and when $f(v)$ is made up of several mutually displaced Gaussians the integration leads to error functions of complex arguments. Tables of the required integral are available (FADDEEVA and TERENTEV, 1954) and a contour plot of the integral is shown by the heart-shaped curves of Fig. 1.

Complex roots w of (3) arise only provided $f(v)$ possesses a minimum. In the case of two hot streams there must be a definite dip between the two peaks of $f(v)$, say at some velocity v_0, where $f'(v_0) = 0$. This is not a sufficient condition. It is also necessary that $\int (v - v_0)^{-1} f'(v)\,dv$ should be positive. For two Gaussians this means the mutual drift must exceed a value comparable with $(KT/m)^{1/2}$.

Broadly speaking, when the two streams are too hot, LANDAU damping defeats two-stream growth. There exists a definite threshold for instability. Close to the Maxwellian distribution, i.e. with only small mutual drift, there is stability. Certainly all single-humped distributions are stable. Only sufficiently drastic departures from the Maxwellian state cause instability.

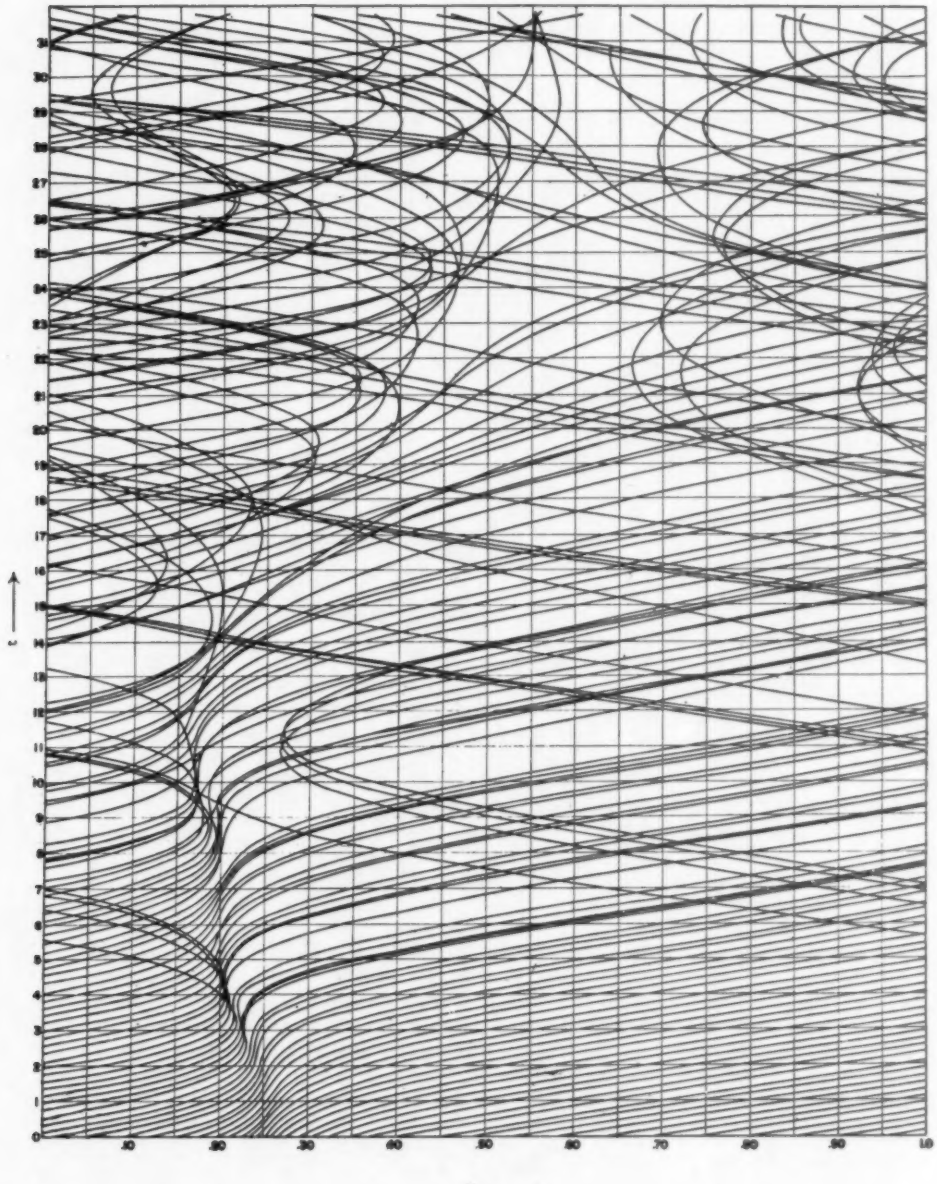


FIG. 2.—Time-displacement graphs of 32 electron sheets (out of 256 computed). t, in units of ω_{pe}^{-1}, vs. x, in units of the self-excited space period L.

Two colliding hydrogen plasmas become unstable when the velocity of approach exceeds the mean internal electron velocity. This corresponds to Mach number 43. For slower drifts, there is no ion-ion instability in the presence of the electron unless the ions are about $3\frac{1}{2}$ times colder than the electrons.

7. THE BUILD-UP

The background noise which triggers off the growth of instabilities depends upon the history of the system, i.e. the reasons for having a peculiarly non-Maxwellian distribution. In the case of interpenetrating beams some of the thermal energy of each beam resides in plasma modes which, on interpenetration, become unstable.

Initially, there is only one KT in each mode and only a small fraction of all modes becomes unstable. It may take many powers of e before amplification has raised the energy in these modes to the level of the mutual drift energy (at the cost of the latter). Some 30 plasma periods may elapse in typical cases. This shows up the importance of a linear, small-signal theory: only

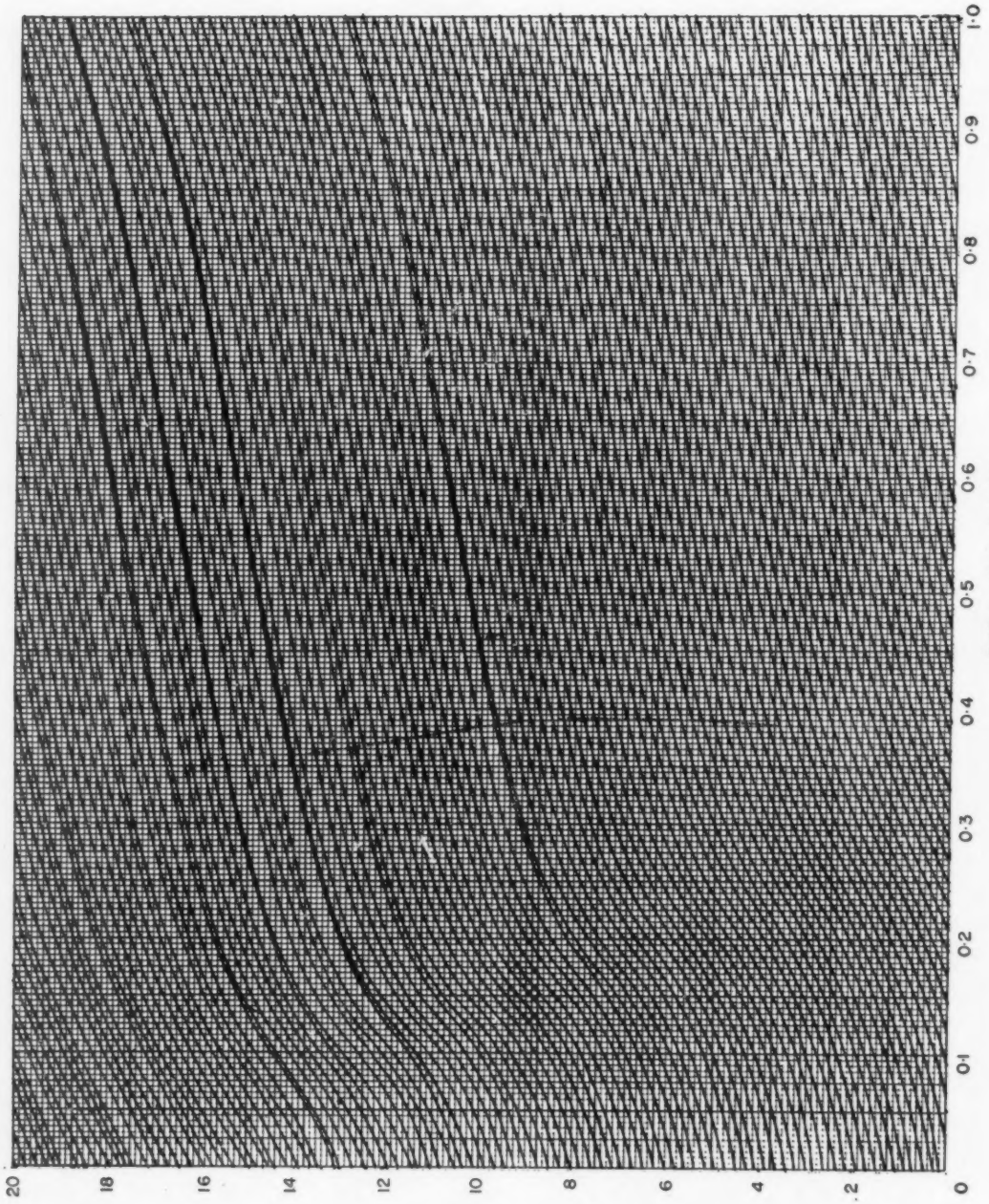


FIG. 3.—Graphs of 32 electron sheets (out of 256) as in Fig. 2, but with initial excitation at lower amplitude. Units as in Fig. 2. t from 0 to 20 (about 3 electron plasma periods).

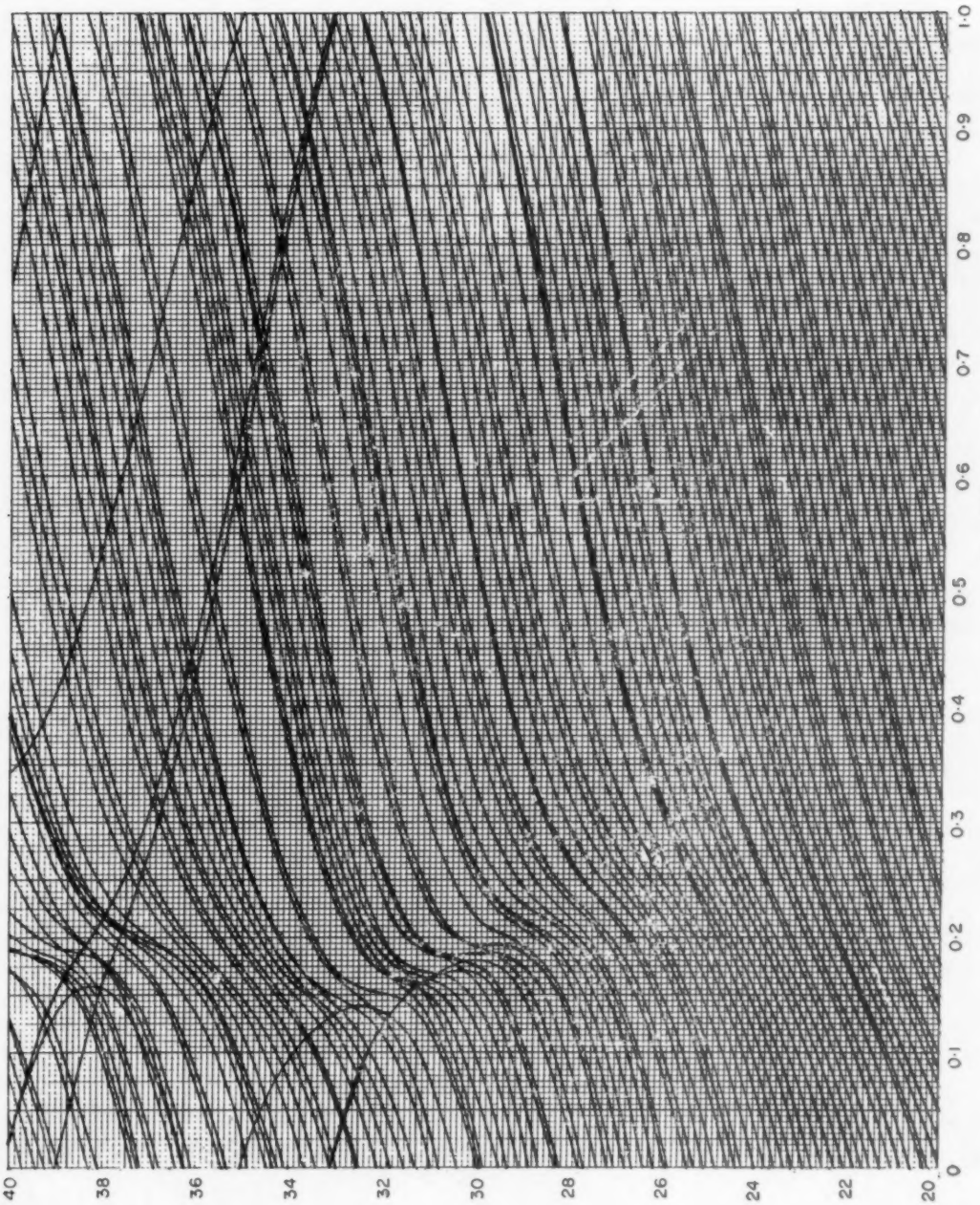


FIG. 4.—Continuation of Fig. 3, t from 20 to 40.

OL.
2
961

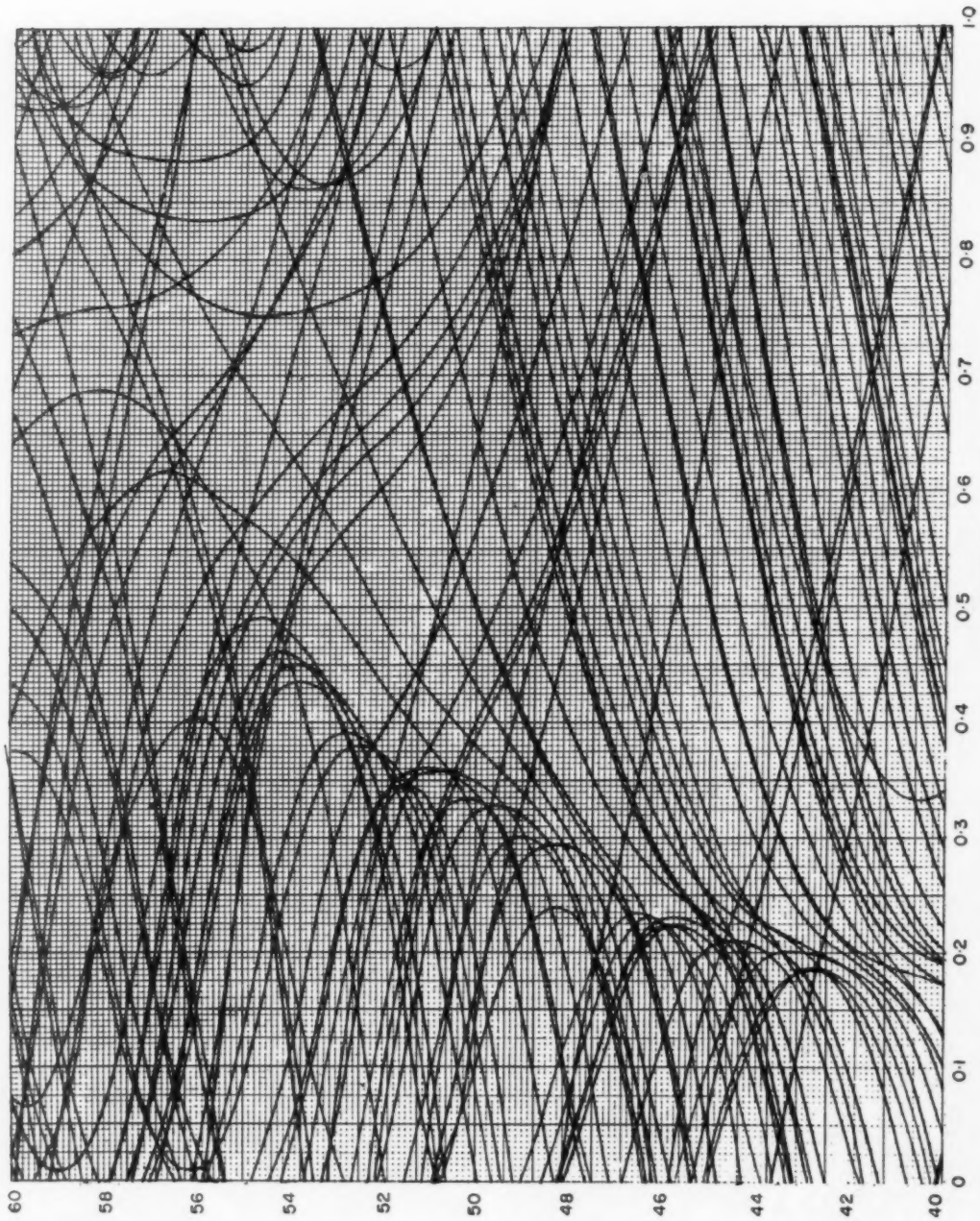


FIG. 5.—Continuation of Fig. 4, t from 40 to 60.

VOL
2
196

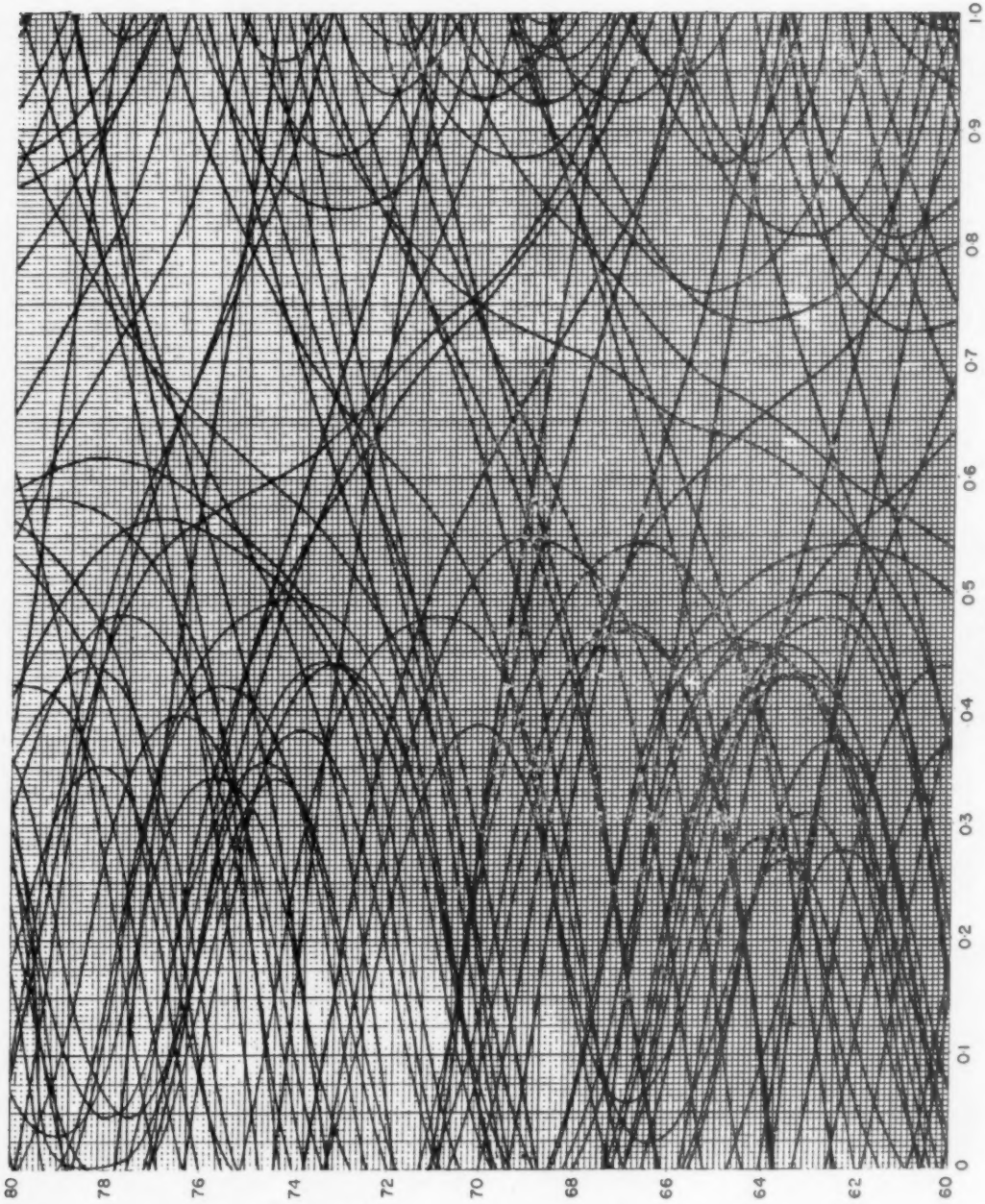


FIG. 6.—Continuation of Fig. 5, t from 60 to 80.

VOL.
2
961

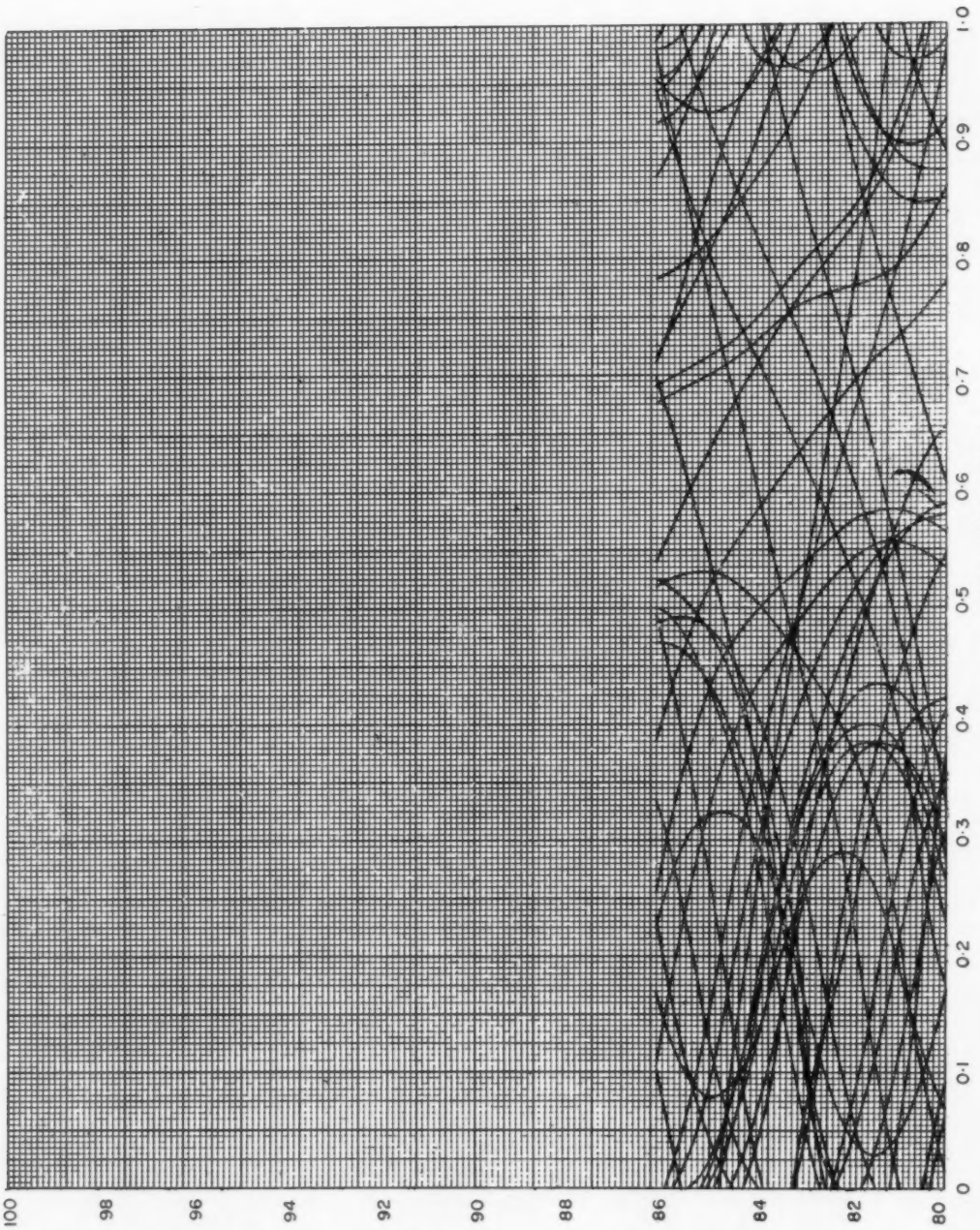


FIG. 7.—Continuation of Fig. 6, t from 80 to 86.

VOL
2
196

during the last two or three plasma periods would non-linear terms become important.

The growth rate, given by the imaginary part of ω, is usually comparable with ω_p. The real part is of similar magnitude and often smaller, so that there is precious little oscillation in the exponential build-up. The two-stream mechanism excites longitudinal electric fields, but the direction of these fields need not coincide with that of mutual streaming.

8. NON-LINEAR PHENOMENA

The excitation of wave vectors of different direction alone guarantees that disturbances become randomized through non-linear interaction (STURROCK, 1957). But one-dimensional excitation, or excitation of a single k-vector, also leads to non-linear effects in due course and these one-dimensional non-linear phenomena can be studied by numerical methods up to arbitrarily high level.

Figure 2 (from BUNEMAN, 1959) shows the result of such a calculation for the electron-ion system with large initial mutual drift. 256 electron sheets and 256 ion sheets were tracked through their combined instantaneous fields from the configuration predicted by linear theory into the non-linear regime, for about 5 plasma periods. Time-displacement graphs of 32 of the electron sheets are presented graphically in Fig. 2 (a self-excited periodicity in space is preserved throughout: the pattern repeats on the right and left).

The transition from order to chaos is surprisingly rapid. The first break-up of order is actually caused by electrons bouncing off a wall created by other electrons, not off the ions. The latter have, up to that stage, only acted as a stimulant for electron bunching. Later, there is a tendency for electrons to be trapped into the layers where ions congregate, thus restoring neutrality. But other electrons continue to streak right across both forward and backward.

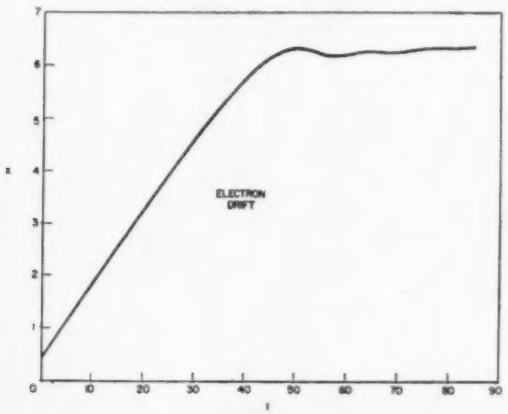


FIG. 8.—Displacement of the *mean* position of the original 256 electrons.
Units as in Figs. 2–7, but x plotted vertically, t horizontally.
Electron drift stops at $t = 50$.

This type of calculation was extended by Vaughan-Williams at A.W.R.E., Aldermaston, England and some minor corrections were incorporated. Figs. 3–7 show the electrons and confirm all the qualitative features of the first calculations. Fig. 8 shows the mean position of all the electrons: they bounce off the ions in bulk at $t = 50$. This is a 'collective collision': the ions have picked up the momentum from the electrons and the total electric current reverses just before that instant. Figs. 9–13 show the ions, eventually brought to mutual interpenetration and disorder, like the electrons.

9. CREATION OF A MAXWELLIAN DISTRIBUTION

Histograms of the eventual electron and ion velocities are shown in Figs. 14 and 15. They are typically Maxwellian. The two species have each acquired a high temperature: the total energy (which like the total momentum is conserved throughout the calculation) stems from the initial streaming motion of the electrons alone. The two species were 'cold' initially, except to the extent that rounding-off errors represent a little noise and heat.

Figure 16 shows a phase-space diagram, v vs. x, of the electrons. The continuous line gives the initial ordered distribution, the scattered points the final 'random' distribution. Each electron sheet is represented by a speck of standard size and the total area covered by the specks is the same initially as finally, in accordance with Liouville's theorem. Nevertheless, the final distribution is much more appropriately described as a dilute distribution covering a large phase-space area: we have changed the entropy.

Unfortunately, the numbers of sheets handled in this calculation are not large enough to allow observation of significant departures from a Maxwellian distribution. For instance, one might look for an excessive 'tail' representing runaway particles. Calculations with 1024 sheets of each species are in progress.

10. GENERALIZATIONS

The calculations are fully representative of one-dimensional, electron-ion, two-stream interaction: the only specific number fed into the programme was the electron-ion ratio: the hydrogen ratio was chosen.

Since the two-or three-dimensional problem is beyond the capacity of even the largest machines, we can only speculate what will happen generally between interstreaming electrons and ions: The added degrees of freedom are likely to accelerate the process of randomization.

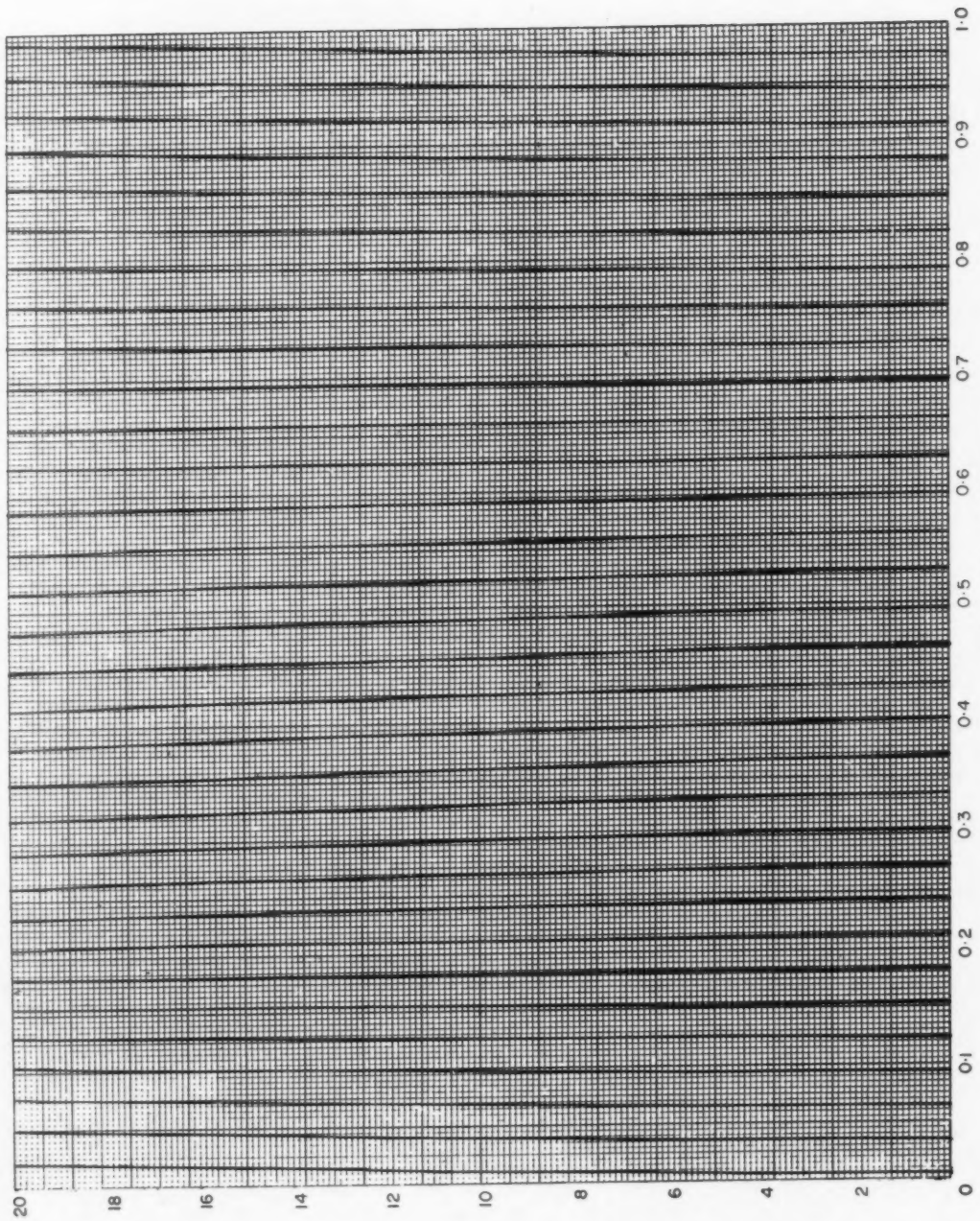


FIG. 9.—Graphs of 32 ion sheets (out of 256 computed), proceeding simultaneously with the electrons illustrated in Fig. 3.

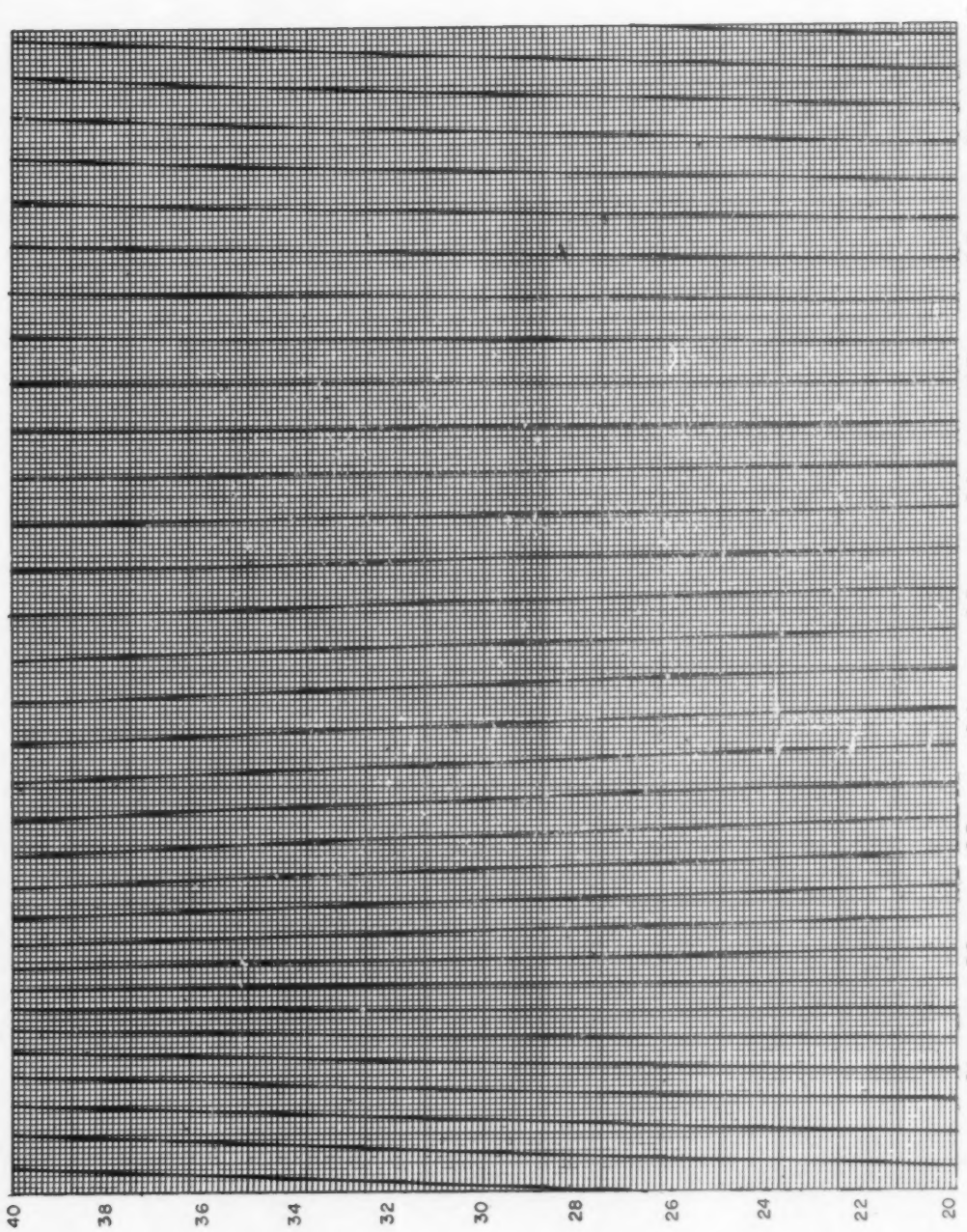


FIG. 10.—Ion motion accompanying electrons shown in Fig. 4.

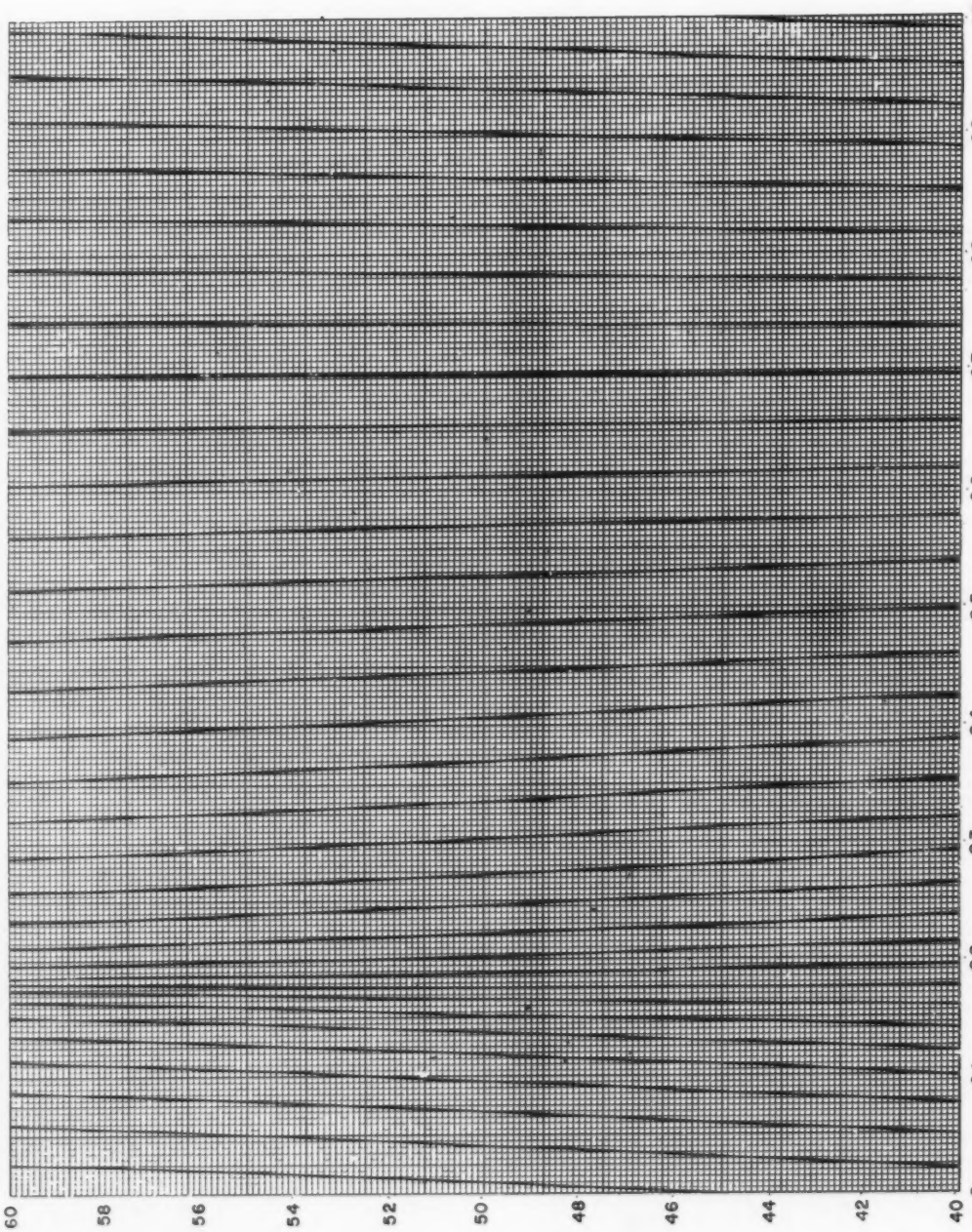


FIG. 11.—Ion motion accompanying electrons shown in Fig. 5.

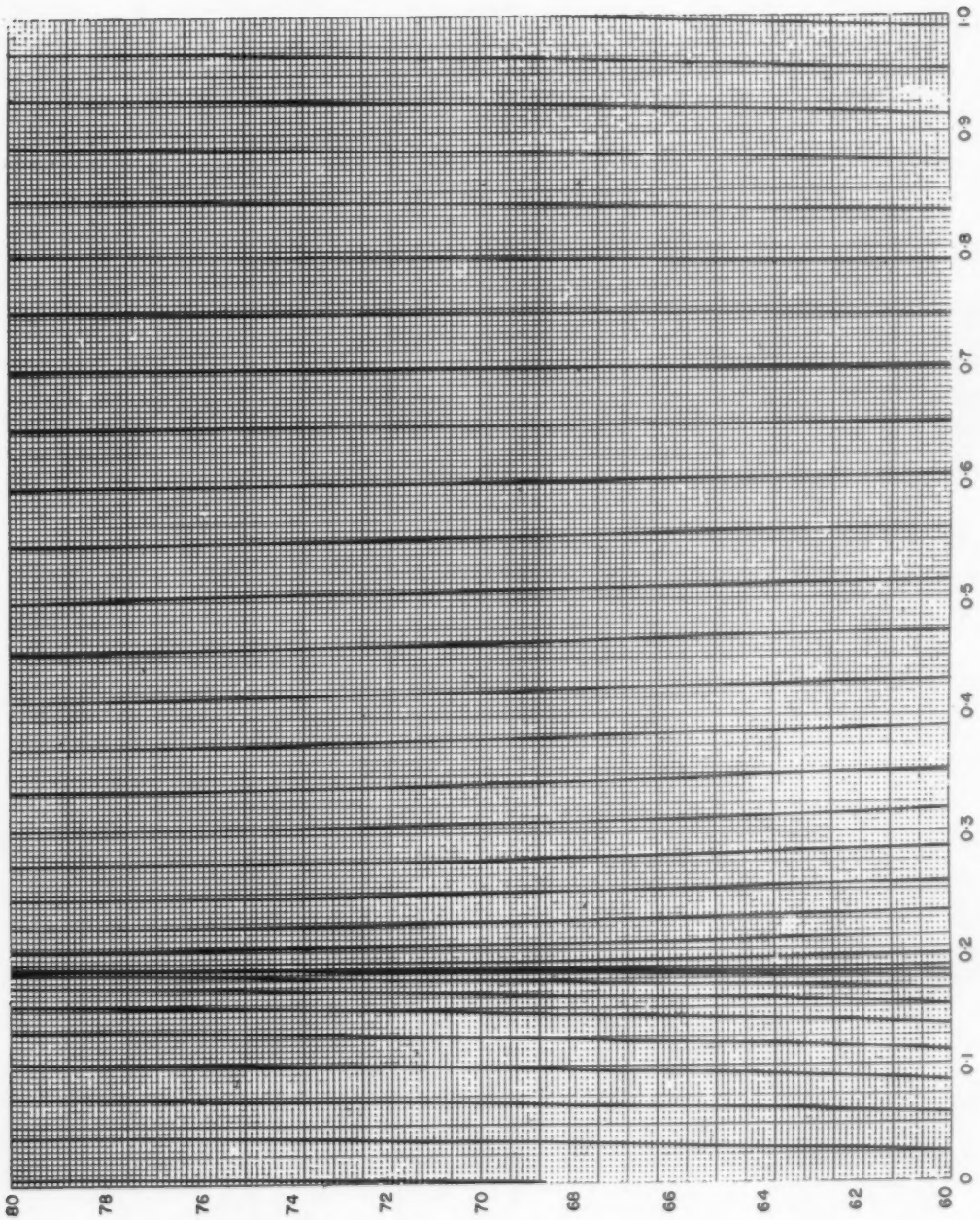


FIG. 12.—Ion motion accompanying electrons shown in Fig. 6.

OL.
2
961

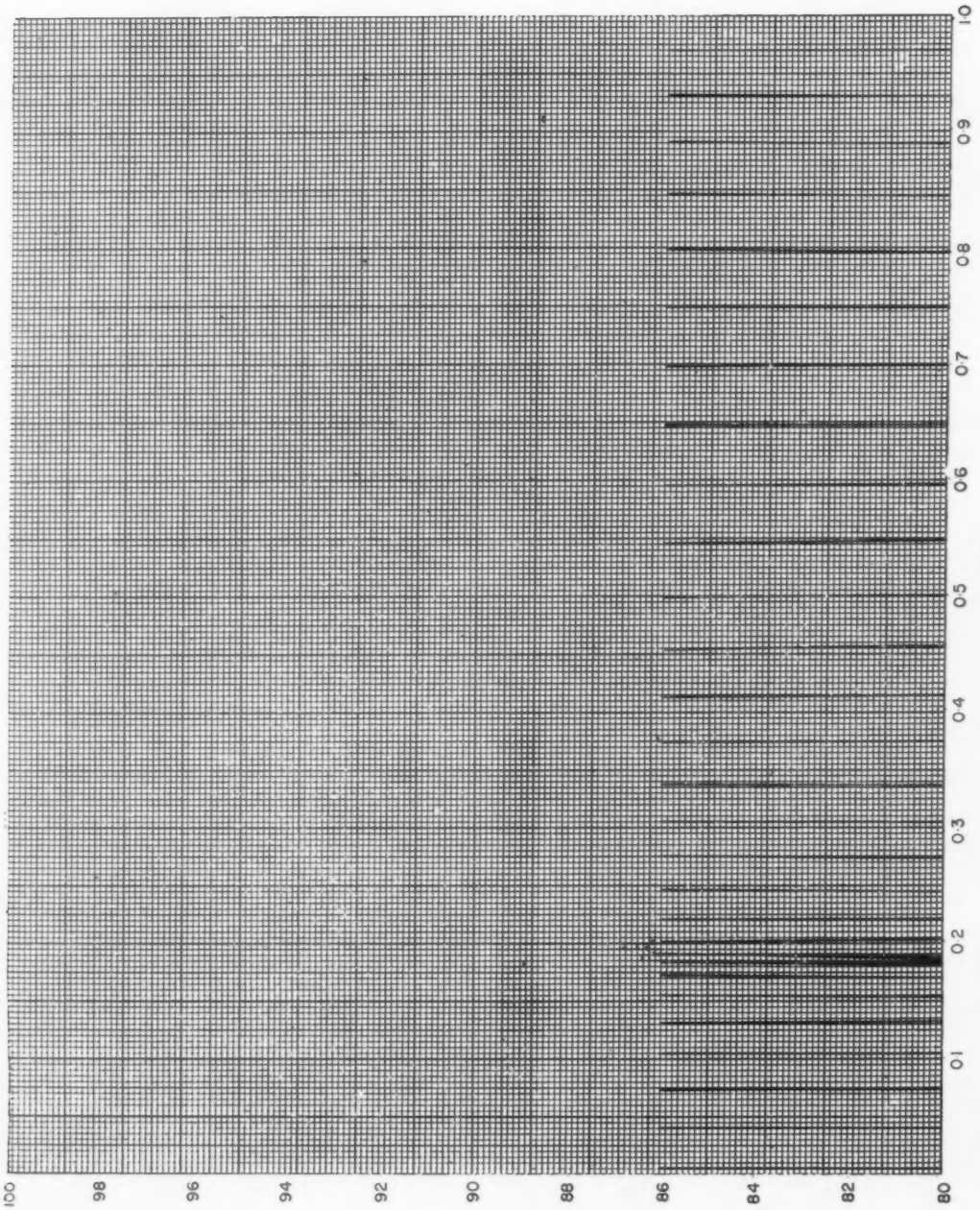


FIG. 13.—Ion motion accompanying electrons shown in Fig. 7.

VOL
2
196

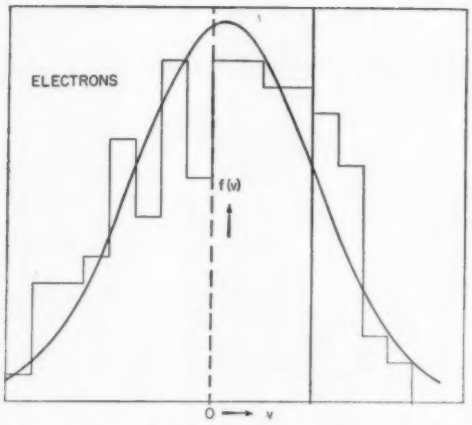


FIG. 14.—Histogram of eventual electron velocities. Continuous curve is best-fitting Gaussian.
Solid vertical line on right shows initial delta-function velocity distribution.

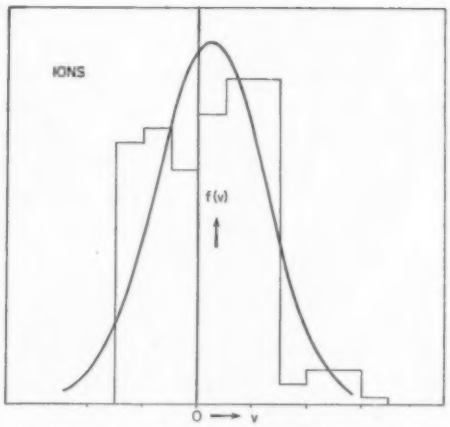


FIG. 15.—Histogram of eventual ion velocities. Continuous curve is best-fitting Gaussian.
Solid vertical line shows initial delta-function distribution (rest). Horizontal scale differs from that in Fig. 14. Displacement of Gaussian peak to the right due to ions absorbing the electrons' initial momentum.

VOL.
2
1961

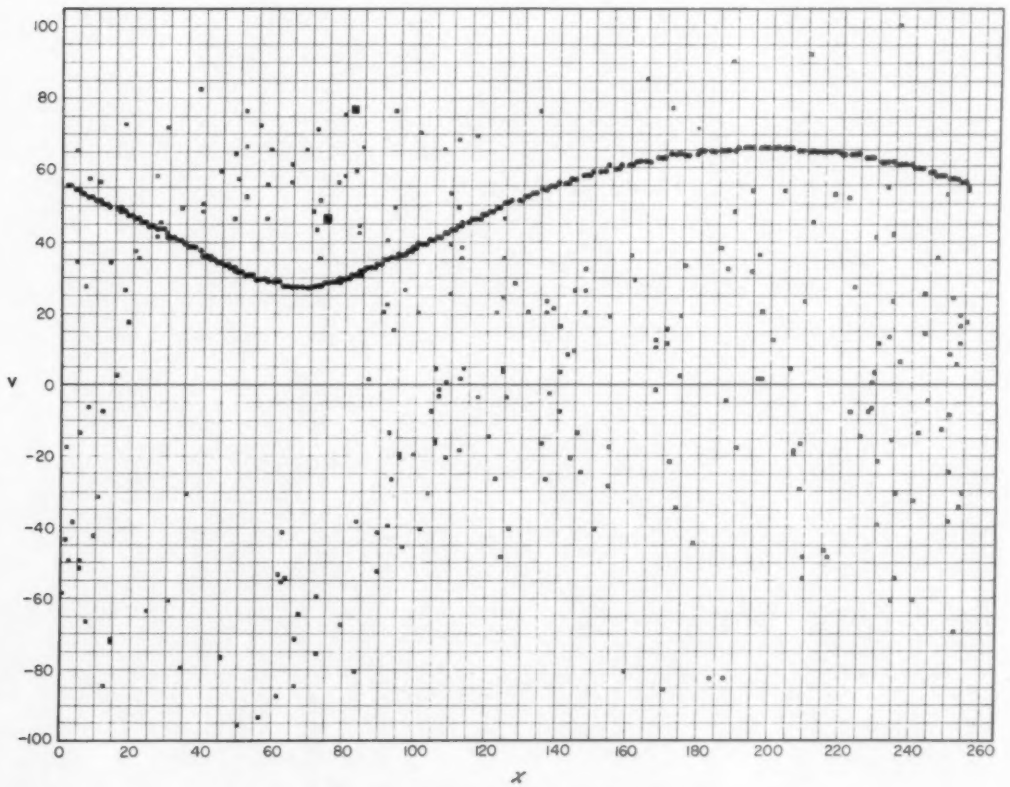


FIG. 16.—Phase space of electrons, v vs. x.
Almost continuous sinusoidal line shows Boltzmann distribution at $t = 0$ (end of 'linear' build-up of instability, start of numerical calculation), scattered points show Boltzmann distribution at $t = 86$. Initial and final set of specks cover same area.

No transverse magnetic field was assumed: the linear theory shows that such a field would impede build-up slightly. Time-scales will expand like the third root of the magnetic field. (This throws an interesting light on diffusion rates across the magnetic field by collective processes.)

There is precious little doubt, then, that two-stream or multi-stream instability will return a system to something like a Maxwellian distribution quickly. Collective effects act like turbulence in a fluid dynamics.

However, the initial non-Maxwellian states covered by theory and calculation so far are all completely uniform in space. No rigorous perturbation analysis (from the collision-free Boltzmann equation) of a self-consistent non-uniform plasma seems to exist. This is most urgently required. It is projected to investigate at Cambridge the stability of a pinched plasma sheet (without using hydromagnetic approximations) and to see whether collective effects will blow up this sheet, quickly creating monotonous Maxwellian uniformity throughout space, or whether true confinement is possible.

REFERENCES

ALLIS W. P. (1958) *Proc. Symposium on Electronic Waveguides*, Polytechnic Inst. Brooklyn, p. 149, Interscience, New York.
BERNSTEIN I. B. (1958) *Phys. Rev.* **109,** 10.
BERNSTEIN I. B., GREENE J. M. and KRUSKAL M. D. (1957) *Phys. Rev.* **108,** 546.
BOHM D. and GROSS E. P. (1949) *Phys. Rev.* **75,** 1851.
BUNEMAN O. (1958) *Phys. Rev.* **112,** 1504.
BUNEMAN O. (1959) *Phys. Rev.* **115,** 503.
FADDEEVA and TERENTEV (1954) *Probability Integral*, Gostekhizdat, Moscow.
KAMPEN VAN (1955) *Physica s'Grav.* **21,** 949.
LANDAU L. J. (1946) *J. Phys. U.S.S.R.* **10,** 25.
PARKER E. N. To be published.
STURROCK P. A. (1957) *Proc. Roy. Soc.* **A242,** 277.

VOL
2
196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 135 to 137. Pergamon Press Ltd. Printed in Northern Ireland

INSTABILITIES AND GROWING WAVES IN ELECTRON BEAM DEVICES

J. R. PIERCE
Bell Telephone Laboratories, Inc., Murray Hill, New Jersey

Abstract—Some forms of electron flow are unequivocally unstable in the sense that perturbations grow with time. Excessive currents between grids or through tubes in the presence or absence of neutralizing positive charge provide examples. Such instabilities are not associated with wave components which grow with distance for real frequencies. Thin beams in magnetic fields also show instability under the action of space charge forces. Tubes sometimes exhibit gain in the absence of any wave component which exhibits spatial growth for real frequencies.

There is a large class of amplifiers whose operation can be explained in terms of spatially growing waves. The physical mechanism can be variously described as instability in a moving co-ordinate system, as a result of a negative dielectric constant at the frequency of operation which causes electrons to attract rather than to repel one another, as a result of coupling between positive energy and negative energy waves, and in other terms.

In the easitron, an inductive wall results in growing waves. In the resistive wall amplifier, the abstraction of power from negative energy waves results in growth. In the travelling-wave tube amplifier, a positive power circuit wave is coupled to a negative power space charge wave. In the double stream amplifier, the negative power space charge wave of the faster stream is coupled to the positive power space charge wave of the slower stream. In velocity jump and rippled stream amplifiers, periodic discontinuities couple the positive power and negative power waves of the same stream. In wave-type parametric amplifiers, moving periodic discontinuities couple two unattenuated waves.

The explanations listed are convenient in connexion with the particular device, rather than unique.

VOL. 2 1961

SUPPOSE that, as shown in Fig. 1, we inject an electron current of density J_0, composed of electrons with a velocity u_0 specified by a potential V_0, into the space between two parallel short-circuited grids. Suppose we gradually increase the current density. The potential in the space between the grids will gradually fall and, then, as a limiting current density is reached, it will drop abruptly to zero, causing the return of some electrons. The total current density which can pass between the grids decreases as the distance L between the grids is increased. The current density is (FAY, SAMUEL and SHOCKLEY, 1939):

$$J_0 = 18{\cdot}6 \times 16^{-6} V_0^{3/2}/L^2. \quad (1)$$

Here the units are amperes per square centimetre, volts and centimetres.

The mechanism seems clear. An increase in charge density decreases the potential, and a decrease in potential increases the charge density. Above some charge density, the process runs away. One can in fact apply the equations of LLEWELLYN and PETERSON (1944) to determine the stability of disturbances and one indeed finds that for currents just above the limiting value there is an exponentially increasing disturbance (PIERCE, 1944a).

One might think that the presence of a fixed positive charge just neutralizing the average electron charge ('space-charge neutralization') would avoid such an instability, but it merely increases the limiting stable current density (PIERCE, 1944b) to

$$J_0 = 104 \times 10^{-6} V_0^{3/2}/L^2. \quad (2)$$

A limiting stable current (not current density), I_0, can also be found for a long space-charge neutralized beam, (PIERCE, 1944b),

$$I_0 = 190 \times 10^{-6} V_0^{3/2} \quad (3)$$

but experiment indicates that the attainable current actually corresponds very closely to that calculated in the absence of ions,

$$I_0 = 29{\cdot}3 \times 10^{-6} V_0^{3/2}. \quad (4)$$

When the positive charge consists of real ions which can move, oscillations can occur (PIERCE, 1948) at

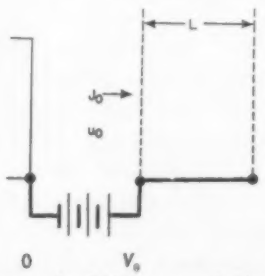


FIG. 1.—Electrons of velocity u_0 constituting a current J_0 injected into the space between two grids a distance L apart.

currents lower than that given by (3), these may reduce the ion density. An alternative to the constant potential assumed in deriving (3) is a sequence of virtual cathodes (PIERCE, 1944b).

It is interesting to note that the limiting number density of electrons corresponding to (3) or (4) can be obtained by dividing the current I_0 by the electron velocity u_0 times the electron charge e times the area of the tube A. This leads to

$$n = 2{\cdot}00 \times 10^7 \, V_0/A \qquad (3a)$$

$$n = 3{\cdot}08 \times 10^6 \, V_0/A. \qquad (4a)$$

What the implications of this may be for electrons or ions with a velocity distribution moving in the tube of a fusion machine I would not care to guess. We may note that the number densities will be the same in the case of singly charged ions.

This instability in the presence of ions is interesting in that the space-charge waves in a neutralized moving beam of electrons are unattenuated for real frequencies. Thus, one need not have spatially growing waves at real frequencies or temporally growing waves for real phase constants in order to have temporally growing disturbances in the overall system. The klystron amplifier demonstrates that one does not need spatially growing waves at real frequencies in order to amplify a signal.

The instability of the diode at high currents shows the importance of boundary conditions. We can transform into any set of moving co-ordinates we wish in discussing the waves in electron flow, but the stability of such flow depends strikingly on the speed of the electrons with respect to the grids and the distance between the grids.

On the other hand, spatially 'increasing' solutions at real frequencies do not always lead to gain. To see this we need merely note exponentially 'spatially growing' solutions in waveguides at frequencies below cut-off frequency.

One way to avoid drawing incorrect conclusions from a superficial examination of particular wave-type solutions is to solve the overall problem posed by the physical system. This can be done by fitting the boundary conditions by means of a collection of special solutions or waves (PIERCE, 1944b), or it can be done by a transform method (WALKER, 1953).

It has been noted that in some cases an increasing oscillation (WALKER, 1953) or a spatial gain in power (in the klystron) exists in the absence of any waves which increase spatially at real frequencies. In a number of electron beam devices, however, both theory and experiment indicate that the amount of gain is very closely related to the increase along the length of the device of an increasing wave component of the overall disturbance.

Sometimes this increasing wave is of a rather special type. Thus, a thin hollow cylindrical beam of electrons in a longitudinal magnetic field exhibits growing waves at frequencies below a limiting frequency and the growth of these disturbances can be explained by a very elementary consideration of the forces acting on line charges parallel to a uniform magnetic field (PIERCE, 1956).

Another special type of growth is exhibited in the 'easitron,' in which an electron beam is surrounded by a sequence of resonators which are not coupled to one another. Growing waves occur at frequencies such that the impedance of the resonators is inductive. We can explain this by saying that in ordinary space the dielectric constant is positive and electrons repel one another. For electrons surrounded by resonators, at certain frequencies bunches of electrons attract one another and the bunching of the electron stream increases as the electrons travel along. A cloud of ions can serve as a 'resonator' in the absence or presence of a longitudinal magnetic field (PIERCE, 1948).

Indeed, we can note that in the travelling-wave tube and related devices the electrons see an inductive impedance in a growing wave. It is, however, convenient to think of the growing waves in travelling-wave tubes and related devices in somewhat different terms.

The growing waves which account for the gain of many electron beam devices can be explained in terms of a negative-power space-charge or plasma wave on a moving stream of electrons. Such a stream characteristically exhibits a pair of space-charge waves. One is a *slow wave* whose phase velocity is less than the electron velocity. The other is a *fast wave* whose phase velocity is greater than the electron velocity. The slow wave has negative power because in setting it up the electrons have to be bunched together in regions of less than average velocity. In the fast wave the electrons are bunched together in regions of greater than average velocity, and the power is positive. For both waves the group velocity is in the direction of electron motion.

When an electron beam is surrounded by an electrically lossy tube, the slow wave continually supplies energy to the lossy tube and hence it grows in amplitude. This *resistive wall amplifier* is perhaps the most direct exhibition of the negative power of the slow space-charge wave.

Growing waves are also encountered when an electron stream is shot through a cloud of ions (PIERCE, 1948), near a second electron stream (HAEFF, 1948;

VOL 2 196

PIERCE and HEBENSTREIT, 1949), or near a circuit which supports an electromagnetic wave (KOMPFNER, 1946).

In the double-stream amplifier the increasing wave results from coupling between the slow negative-power wave of the faster stream and the fast, positive-power wave of the slower stream. An increasing and a decreasing wave result; both have the same phase velocity. For each, the sum of the negative power of the slow-wave component and the positive power of the fast-wave component is zero so that the overall increasing or decreasing disturbance has no power.

In the travelling-wave tube the slow, negative-power space-charge wave is coupled to an electromagnetic wave which has positive power. Again, the growing and decreasing waves have no power. At the output end the electromagnetic wave is abstracted as output. The emerging electron stream supports a large negative-power space-charge wave, so that its kinetic power is less than the kinetic power of the undisturbed electron flow.

The slow space-charge wave of an electron stream can be coupled to the fast space-charge wave of the same stream by periodic variations of the potential or diameter of the structure surrounding the beam, or by periodic variations of the beam diameter (FIELD, TIEN and WATKINS, 1951; TIEN and FIELD, 1954; BIRDSALL, 1954). This also results in zero-power increasing and decreasing waves.

It is of interest to consider some aspects of these growing waves. It has sometimes been implied that growing waves grow because energy is added to them along their length. The growing waves we have considered have zero energy. Power can be derived from them by abstracting or destroying part or all of a positive-energy component of the wave, as the electromagnetic circuit component of the growing wave in a travelling-wave tube.

It is also interesting to note that the growing waves in double-stream amplifiers, rippled-stream or velocity jump amplifiers and travelling-wave tubes are made up of unattenuated component waves having group velocities in the same direction. Thus each component wave can be set up at the cathode-end of the electron stream, and when the components are coupled together they will interact so that the disturbance grows in the direction of electron motion.

In the backward-wave amplifier or oscillator (WALKER, 1953) the slow space-charge wave of the electron stream interacts with an electromagnetic wave having a group velocity opposite to the electron flow; the interaction is attained because the wave has a field component with a phase velocity in the direction of electron flow. Both interacting components have negative power in the direction of electron flow and hence they couple to give unattenuated waves. Thus, the gain of the backward-wave amplifier is not due to an increasing wave. The gain occurs because the a.c. circuit voltages of the two unattenuated waves involved add at the output (cathode) end and nearly cancel at the input (collector) end of the tube. In the backward-wave oscillator, the circuit voltages exactly cancel at the collector end of the circuit.

We have seen that in the rippled-beam or velocity-jump tube, periodic features of the beam or of its environment can result in gain. A *moving* feature which partially reflects a wave can transfer energy to the wave (PIERCE, 1948). The reflecting feature could be an electric field, a space-charge or plasma wave, a sound wave or a cloud of particles. A sequence of such reflecting features can be so spaced that their effects add in phase (PIERCE, 1959). Amplification of waves caused by such a periodic sequence of moving reflectors is an example of *parametric amplification.*

REFERENCES

BIRDSALL C. K. (1954) *Proc. I.R.E.* **42,** 1628.

FAY C. E., SAMUEL A. L. and SHOCKLEY W. (1939) *Bell Syst. Tech. J.* **17,** 49.

FIELD L. M., TIEN P. K. and WATKINS D. A. (1951) *Proc. I.R.E.* **39,** 194.

HAEFF A. V. (1948) *Phys. Rev.* **79,** 1523.

KOMPFNER R. (1946) *Wireless World* **52,** 369.

LLEWELLYN F. B. and PETERSON L. C. (1944) *Proc. I.R.E.* **32,** 144.

PIERCE J. R. (1944a) *Phys. Rev.* **66,** 29.

PIERCE J. R. (1944b) *J. Appl. Phys.* **15,** 721.

PIERCE J. R. (1948) *J. Appl. Phys.* **19,** 231.

PIERCE J. R. and HEBENSTREIT W. B. (1949) *Bell Syst. Tech. J.* **28,** 33.

PIERCE J. R. (1956) *I.R.E. Trans.* **E-D.3,** 183.

PIERCE J. R. (1959) *J. Appl. Phys.* **30,** 1341.

TIEN P. K. and FIELD L. M. (1954) *Proc. I.R.E.* **40,** 694.

WALKER L. R. (1953) *J. Appl. Phys.* **24,** 854.

VOL. 2 1961

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 138 to 145. Pergamon Press Ltd. Printed in Northern Ireland

PLASMA INSTABILITIES ASSOCIATED WITH ANISOTROPIC VELOCITY DISTRIBUTIONS

E. G. HARRIS
Oak Ridge National Laboratory,* Oak Ridge, Tennessee
and
University of Tennessee, Knoxville, Tennessee

Abstract—The general dispersion relation is derived for small amplitude waves in a fully-ionized plasma in an external magnetic field. This derivation is based on the Vlasov equations (Boltzmann equations without collision terms for the electrons and ions plus Maxwell's equations). The dispersion relation involves integrals over the zero-th order velocity distributions. It is found that for sufficiently anisotropic velocity distributions waves exist which have exponentially growing amplitudes. A number of special cases are discussed. It is shown that streams of charged particles passing through a plasma may excite either longitudinal or transverse waves. Other instabilities exist when the distributions are such that the particles have their velocity vectors perpendicular to the magnetic field.

1. INTRODUCTION

IN recent years attempts to achieve thermonuclear reactions have yielded experimental evidence which suggest the presence of new types of instabilities. Thus, in a recent paper on the pinch effect COLGATE and FURTH (1958) have reported evidence that small-scale turbulence exists within the stabilized pinch configuration and that this turbulence is responsible for decreasing the plasma conductivity, accelerating a small number of particles to high energies and greatly increasing the rate of heat transfer to the walls. The instability does not seem to be of the sort predicted by the hydromagnetic equations which involve a gross motion of the plasma to the walls of the container. These hydromagnetic instabilities are by now fairly well understood. COLGATE and FURTH suggest that the turbulence is due to plasma waves which are excited by runaway electrons.

BERNSTEIN *et al.* (1958a and 1958b) have reported a number of peculiar phenomena observed in Stellarator discharges. The phenomena include a decay of the discharge current in abrupt steps, the generation of intense non-thermal microwave noise and bursts of X-ray due to loss of confinement of runaway electrons. These phenomena occurred under conditions for which the plasma should not be subject to hydromagnetic instabilities.

The instabilities mentioned above bear some resemblance to an instability reported by ALFVÉN *et al.* (1948). This instability was found in experiments on trochotrons. The main results may be summarized as follows. Electrons which are emitted in crossed electric and magnetic fields move in trochoidal paths and constitute a beam perpendicular to both fields. At low emission, i.e. at low electron density, the motion is in accordance with the motion calculated for single particles in external fields. At higher densities, however, the energy distribution of the electrons is rapidly changed in such a way that the electrons reach electrodes which are negative with respect to the cathode. At the same time the beam exhibits an abnormally strong noise. The transition between the normal and abnormal operating conditions occurs very sharply as the emission is increased. MALMFORS (1950) attempted to explain these results on the basis of unstable plasma oscillations. We shall review the theoretical work on this problem later in this paper.

Most investigations of a plasma in a magnetic field have relied on hydrodynamic equations, i.e. equations relating the densities, average velocities, pressures and temperatures of the electrons and ions. In very high temperature and low density plasmas large departures from local thermodynamic equilibrium may be expected and the validity of hydrodynamic equations is questionable. Indeed, the calculations of BERNSTEIN (1958) indicate that a much wider variety of waves can exist in a plasma than the hydrodynamic equations predict. BERNSTEIN's calculations are based on the Vlasov equations (i.e. Boltzmann equations without collision terms for the electrons and ions plus Maxwell's equations). It seems a reasonable guess that some of the experimentally observed instabilities which seem incomprehensible on the basis of the usual theories can be explained on the basis of the Vlasov equations. The present paper is devoted to exploring this possibility.

* Operated by Union Carbide Corporation for the U.S. Atomic Energy Commission.

The development of this work is the following. In Section 2 we present the general formulation of the linearized problem. In doing so we follow fairly closely the work of BERNSTEIN (1958). Unlike BERNSTEIN we do not choose the zero-th order distribution functions to be Maxwellian. The principal result of Section 2 is the dispersion relation connecting the frequency and the wave vector **k** for waves in the plasma. This dispersion relation involves integrals over the zero-th order distribution functions which at this stage of the work are unspecified. In the following sections various special cases of the dispersion relation are considered with special attention to those zero-th order distribution functions which lead to instabilities.

2. THE LINEARIZED VLASOV EQUATIONS

The following set of equations is used:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{e}{M}\left(E + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \tag{1}$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi, \tag{2}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{3}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 A = \frac{4\pi}{c}\Sigma e \int \mathbf{v} f \, d^3v, \tag{4}$$

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{dt^2} - \nabla^2\Phi = 4\pi\Sigma e \int f \, d^3v, \tag{5}$$

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0. \tag{6}$$

In the above $f(\mathbf{r}, \mathbf{v}, t)$ is the distribution function for the species of particles (ion or electron) whose charge and mass are e and M. There will be an equation such as equation (1) with appropriate values of e and M for each species of particle. Unless otherwise noted summation signs will refer to summation over species, and to simplify the notation we shall not use indices on f, M, and e. **E** and **B** are the electric and magnetic fields and are derived in the usual way from the potentials **A** and Φ. Equation (6) is the Lorentz gauge condition and is not independent of the preceding equations. It may be derived from equations (1), (4) and (5).

We now consider systems which depart only slightly from an equilibrium configuration in which $\mathbf{E} = 0$ and $f = f_0(\mathbf{v})$. The uniform magnetic field is taken to be in the z-direction. It is easily seen that f_0 will satisfy equation (1) if

$$f_0 = f_0(v_\perp, v_z), \tag{7}$$

where

$$v_\perp^2 = v_x^2 + v_y^2. \tag{8}$$

We write

$$f = f_0 + f_1, \tag{9}$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \tag{10}$$

and assume that f_1, **E** and $\mathbf{B}_1$ are small quantities whose squares and products may be neglected. Neglecting these small quantities, we may write equation (1) as:

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} + \frac{e}{Mc}(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_1}{\partial \mathbf{v}} = -\frac{e}{M}\left[\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}_1\right] \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \tag{11}$$

We shall now assume that f_1, **E**, and $\mathbf{B}_1$ have their space and time dependence given by a factor

$$\exp(i\mathbf{k} \cdot \mathbf{r} + i\omega t). \tag{12}$$

Because of this assumption certain improper integrals will arise in our calculations. In evaluating these the prescription of LANDAU must be followed (LANDAU, 1946). In our work this amounts to assuming that ω has a small negative imaginary part. The dispersion relation obtained in this way is the same as would be obtained by the solution of an initial value problem. Equation (11) may now be written

$$i(\omega + \mathbf{k} \cdot \mathbf{v})f_1 + \omega_c\left(v_x \frac{\partial f_1}{\partial v_y} - v_y \frac{\partial f_1}{\partial v_x}\right) = +\frac{ie}{M}\left[\left(\Phi - \frac{\mathbf{v} \cdot \mathbf{A}}{c}\right)\mathbf{k} + \frac{1}{c}(\omega + \mathbf{k} \cdot \mathbf{v})\mathbf{A}\right] \cdot \frac{\partial f_0}{\partial \mathbf{v}} \tag{13}$$

where $\omega_c = \dfrac{eB}{Mc}$ is the cyclotron frequency.*

It is now convenient to introduce the cylindrical co-ordinates $v_\perp$, ϕ, and v_z in velocity space. Also we choose **k** to lie in the xz-plane. Then

$$\mathbf{v} = \mathbf{e}_1 v_\perp \cos\phi + \mathbf{e}_2 v_\perp \sin\phi + \mathbf{e}_3 v_z \tag{14}$$

and

$$\mathbf{k} = \mathbf{e}_1 k_\perp + \mathbf{e}_2 k_z, \tag{15}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are unit vectors along the x, y and z-axes. Equation (13) may now be written as an ordinary first-order differential equation in ϕ,

$$\frac{\partial f_1}{\partial \phi} + i\left(\frac{\omega + \mathbf{k} \cdot \mathbf{v}}{\omega_c}\right) f_1 = \frac{ie}{M\omega_c}\left[\left(\Phi - \frac{\mathbf{v} \cdot \mathbf{A}}{c}\right)\mathbf{k} + \frac{1}{c}(\omega + \mathbf{k} \cdot \mathbf{v})\mathbf{A}\right] \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \tag{16}$$

* Note that the electron cyclotron frequency is a negative quantity.

Bernstein has shown that the solution of equation (16) is

$$f_1 = \int_{\pm\infty}^{\phi} G(\phi, \phi')S(\phi')\, d\phi' \tag{17}$$

where $S(\phi)$ is the right-hand side of equation (16) and

$$G(\phi, \phi') = \exp\left[-i\left(\frac{\omega + k_z v_z}{\omega_c}\right)(\phi - \phi') - \frac{ik_\perp v_\perp}{\omega_c}(\sin\phi - \sin\phi')\right]. \tag{18}$$

The sign in the lower limit of equation (17) is chosen so that $G(\phi, \phi')$ vanishes at the lower limit when ω has a negative imaginary part.

We now define

$$I_1(\phi) = \int_{\pm\infty}^{\phi} G(\phi, \phi') \cos\phi'\, d\phi', \tag{19}$$

$$I_2(\phi) = \int_{\pm\infty}^{\phi} G(\phi, \phi')\, d\phi, \tag{20}$$

$$I_3(\phi) = \int_{\pm\infty}^{\phi} G(\phi, \phi') \sin\phi'\, d\phi'. \tag{21}$$

In terms of these quantities the solution of equation (16) may be written

$$\begin{aligned} f_1 = \frac{ie}{M\omega_c}\Bigg\{&\Phi\left[k_\perp \frac{\partial f_0}{\partial v_\perp} I_1 + k_z \frac{\partial f_0}{\partial v_z} I_2\right] \\ &+ \frac{1}{c} A_x \left[(\omega + k_z v_z)\frac{\partial f_0}{\partial v_\perp} - k_z v_\perp \frac{\partial f_0}{\partial v_z}\right] I_1 \\ &+ \frac{1}{c} A_y \left[(\omega + k_z v_z)\frac{\partial f_0}{\partial v_\perp} - k_z v_\perp \frac{\partial f_0}{\partial v_z}\right] I_3 \\ &+ \frac{1}{c} A_z \left[\omega \frac{\partial f_0}{\partial v_z} I_2 + k_\perp \left(v_\perp \frac{\partial f_0}{\partial v_z} - v_z \frac{\partial f_0}{\partial v_\perp}\right) I_1\right]\Bigg\}. \end{aligned} \tag{22}$$

Note the linear dependence of f_1 on Φ and $\mathbf{A}$.

The solution for f_1 given in equation (22) must now be substituted into equations (4) and (5) which now become

$$\left(-\frac{\omega^2}{c^2} + k^2\right)\Phi = a_{00}\Phi + a_{01}A_x + a_{02}A_y + a_{03}A_z, \tag{23}$$

$$\left(-\frac{\omega^2}{c^2} + k^2\right)A_i = a_{i0}\phi + a_{i1}A_x + a_{i2}A_y + a_{i3}A_z, \tag{24}$$

where $i = x, y, z$. The a_{ij} involve integrals over the zero-th order distribution functions. They may be obtained from inspection of equations (4), (5) and (22). They are listed in the Appendix.

Equations (23) and (24) are a set of four homogeneous algebraic equations for ϕ, A_x, A_y, A_z. The condition that they have a non-trivial solution is that the determinant of the coefficients must vanish. Setting this determinant equal to zero gives a relation which may be written

$$\omega = \omega(k_\perp, k_z) \tag{25}$$

and is termed the dispersion relation. It is in general a very complicated relation and we shall not attempt to write it out in full. We shall consider certain special cases and give special attention to those zero-th order distributions which cause ω to have negative imaginary parts. By equation (12) such waves would grow exponentially with time and thus indicate plasma instabilities.

2. INSTABILITIES WHEN k IS PARALLEL TO $\mathbf{B}_0$

If $\mathbf{k}$ is parallel to $\mathbf{B}_0$, then $k_\perp = 0$ and $k_z = k$. It is easily shown that

$$a_{01} = a_{02} = a_{10} = a_{13} = a_{20} = a_{23} = a_{31} = a_{32} = 0. \tag{26}$$

These are just the coefficients in equations (23) and (24), which couple Φ and A_z to A_x and A_y. We see that there is no longer any coupling between longitudinal and transverse waves. We also find that

$$a_{11} = a_{22}, \tag{27}$$

$$a_{12} = -a_{21}. \tag{28}$$

The dispersion relation can now be factored into the product of three factors. Setting each factor equal to zero gives

$$1 = \Sigma\, \omega_p^2 \int f_0 \frac{d^3v}{(\omega + kv_z)^2} \tag{29}$$

and

$$\omega^2 = k^2c^2 + \Sigma\, \omega_p^2 \int f_0 \left[\frac{\omega + kv_z}{\omega + kv_z \pm \omega_c} + \frac{\frac{1}{2}k^2 v_\perp^2}{(\omega + kv_z \pm \omega_c)^2}\right] d^3v. \tag{30}$$

(One of the factors has a plus sign preceding ω_c and the other has a minus.) In deriving equations (29) and (30) we have performed an integration by parts. We have changed the normalization of f_0 so that now its integral over velocity space is unity. We have let

$$\omega_p = \left(\frac{4\pi Ne^2}{M}\right)^{\frac{1}{2}} = \text{plasma frequency}. \tag{31}$$

The summation is over species of particles and f_0, ω_p, and ω_c will in general be difficult for each species.

Equation (29) is the dispersion relation for longitudinal plasma oscillations along the lines of **B**. It is the same relation as that found for plasma oscillations in the absence of a magnetic field. It predicts instabilities when one stream of particles passes through another. For instance, let

$$f_{0i} = \frac{1}{2\pi v_\perp}\,\delta(v_\perp)\delta(v_z) = \delta(\mathbf{v}) \tag{32}$$

be the ion distribution function and let

$$f_{0e} = \frac{1}{2\pi v_\perp}\,\delta(v_\perp)\delta(v_z - V) \tag{33}$$

be the electron distribution function. Equation (29) becomes

$$1 = \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{(\omega + kV)^2}. \tag{34}$$

Equations (32) and (33) correspond to electrons moving with velocity V through stationary ions. The dispersion relation equation (34) has been discussed by BUNEMAN (1958). It may be shown that there are complex roots for ω if

$$kV < \left[1 + \left(\frac{M_e}{M_i}\right)^{\frac{1}{3}}\right]^{\frac{3}{2}} \omega_{pe}. \tag{35}$$

Similar instabilities are found when two streams of plasma pass through one another. These have been discussed by KAHN (1957 and 1958) and by HARRIS (1959b).

We now turn our attention to equation (30). It may be shown that the waves described by equation (30) are circularly polarized transverse waves. We shall first discuss an instability which was found by DAWSON and BERNSTEIN (1958). We consider a stream of electrons moving with velocity V directed along the magnetic field through a cold plasma. The distribution functions for the ions and electrons of the plasma will both be

$$f_0 = \delta(\mathbf{v}) \tag{36}$$

and the distribution function for the electrons in the beam will be

$$f_0 = \delta(v - V). \tag{37}$$

Substitution into equation (30) gives

$$\omega^2 = k^2c^2 + \frac{\omega_{pi}^2\omega}{\omega \pm \omega_{ci}} + \frac{\omega_{pe}^2\omega}{\omega \pm \omega_{ce}} + \frac{\varepsilon\omega_{pe}^2(\omega + kV)}{(\omega + kV \pm \omega_{ce})}. \tag{38}$$

In equation (38) the subscript i refers to ions, the subscript e refers to electrons and ε is the ratio of the number density of electrons in the beam to the number density of electrons in the plasma. Equation (38) is the dispersion relation found by DAWSON and BERNSTEIN and analysed in their paper. They show that for a range of values of kV equation (38) has complex roots for ω. The root with a negative imaginary part represents a growing wave. They interpret this instability as being due to a kind of resonance between the cyclotron frequency of the electrons and the Doppler shifted frequency of the wave in the beam.

It is interesting to note that a stream of charged particles passing through a plasma can cause either longitudinal or transverse instabilities.

In the two instabilities which we have just discussed we have assumed that there was no thermal spread of velocities. If thermal motions are taken into account the instabilities are modified by LANDAU damping. The instabilities will grow more slowly or if the velocity spreads are great enough they will not grow at all.

We now consider distribution functions of the form

$$f_0 = \frac{1}{\pi^{3/2}\alpha_\perp^2\alpha_z} \exp\left[-v_\perp^2/\alpha_\perp^2 - v_z^2/\alpha_z^2\right]. \tag{39}$$

If $\alpha_\perp^2 = \alpha_z^2 = 2kT/M$, equation (39) is just the Maxwell-Boltzmann distribution function. It has been shown by BERNSTEIN that there are no instabilities if the plasma is in thermal equilibrium. However, there are interesting instabilities of a plasma in a magnetic field if $\alpha_\perp \neq \alpha_z$. (Speaking roughly, we may say that in this case the perpendicular and longitudinal 'temperatures' are different.) For simplicity we shall only consider the case $\alpha_z = 0$ for which the dispersion relation equation (30) takes on the simple form

$$\omega^2 = k^2c^2 + \Sigma\omega_p^2\left[\frac{\omega}{\omega \pm \omega_c} + \frac{\frac{1}{2}k^2\alpha_\perp^2}{(\omega \pm \omega_c)^2}\right]. \tag{40}$$

WEIBEL (1959) has shown that equation (40) has complex roots for ω even when the external magnetic field is zero. This is easily seen by setting $\omega_c = 0$ and solving the resulting quadratic equation for ω^2. The roots are

$$\omega^2 = \tfrac{1}{2}(k^2c^2 + \Sigma\omega_p^2) \pm \left[\left(\frac{k^2c^2 + \Sigma\omega_p^2}{2}\right)^2 + \tfrac{1}{2}\Sigma\omega_p^2k^2\alpha_\perp^2\right]^{\frac{1}{2}}. \tag{41}$$

It is seen that for one of the roots ω^2 is negative and hence ω is a pure imaginary. For small α the imaginary roots become

$$\omega = \pm i\left[\frac{\frac{1}{2}\Sigma\omega_p^2k^2\alpha_\perp^2}{k^2c^2 + \Sigma\omega_p^2}\right]. \tag{42}$$

In the instabilities of WEIBEL the particles give up their kinetic energy to the electromagnetic field. The

physical mechanism responsible for these instabilities has been discussed by FRIED (1959).

We now return to equation (40) with the magnetic field not equal to zero. An instability has been found by ROSENBLUTH (1959) which may be described as a resonance between the frequency of an ALFVÉN wave and the ion cyclotron frequency. A circularly polarized ALFVÉN wave with its electric field vector rotating in the same direction in which an ion rotates and with a frequency about equal to the ion cyclotron frequency can be expected to interact strongly with the ions of the plasma. Presumably it is this strong interaction which permits the ions to give up their energy to the field thereby causing the amplitude of the waves to grow.

If $\alpha_z \neq 0$ all of these instabilities are modified by LANDAU damping. It is interesting to note that the thermal spread of velocities in directions perpendicular to the magnetic field do not contribute to the LANDAU damping. ROSENBLUTH has called attention to the fact that in the ALFVÉN-cyclotron resonance instability the instability occurs for any α_z between zero and $\alpha_\perp$ although the growth rate of the instability rapidly become smaller as α_z approaches $\alpha_\perp$.*

3. LONGITUDINAL OSCILLATIONS

If one considers waves which propagate at an arbitrary angle to the magnetic field so that $k_\perp$ and k_z can take on any values, the dispersion relation becomes much more complicated. In particular the longitudinal and transverse waves are no longer uncoupled. However, it may be seen by examination of the a_{ij}'s that the terms which couple Φ to $\mathbf{A}$ (that is, a_{01}, a_{02} and a_{03}) are smaller than a_{00} by factors like ω/kc or v_t/c where k is either $k_\perp$ or k_z (which are assumed to be of the same order of magnitude) and v_t is a thermal velocity (which may be an average of either $v_\perp$ or v_z). If we assume that these factors are small, so that a_{01}, a_{02} and a_{03} are negligible and if we also neglect ω/c in comparison with k in equation (23), then equation (23) becomes

$$k^2\Phi = a_{00}\Phi \tag{43}$$

and the dispersion relation becomes simply

$$k^2 = a_{00}. \tag{44}$$

This approximation has also been made by GROSS (1951) and by SEN (1952). It is equivalent to using only Poisson's equation

$$\nabla \cdot \mathbf{E} = -\nabla^2\Phi = 4\pi\Sigma e \int f\, d^3v \tag{45}$$

instead of the complete set of Maxwell's equations.

Using the expression for a_{00} given in the Appendix, we can write equation (44) as:

$$1 = \Sigma \frac{\omega_p^2}{k^2} \sum_{n=-\infty}^{+\infty} \int d^3v \left\{ \frac{\omega_c}{v} \frac{\partial f_0}{\partial v_\perp} \frac{n J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{(\omega + k_z v_z + n\omega_c)} + k_z \frac{\partial f_0}{\partial v_z} \frac{J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{(\omega + k_z v_z + n\omega_c)} \right\}. \tag{46}$$

We shall now examine some distribution functions which lead to instabilities. First we consider

$$f_0 = \frac{1}{2\pi}\,\delta(v_z)\,\frac{\delta(v_\perp - V)}{v_\perp}. \tag{47}$$

This describes a plasma in which all the particles of a species have the same speed perpendicular to the magnetic field and no motion along the field lines. V can be different for electrons and ions. Substituting equation (47) into equation (46) gives

$$1 = \Sigma \frac{\omega_p^2}{\omega_c^2} \sum_{n=-\infty}^{+\infty} \left\{ -\left(\frac{k_\perp}{k}\right)^2 \frac{n}{\left(\frac{\omega}{\omega_c} + n\right)} \left[\frac{1}{b}\frac{d}{db} J_n^2(b)\right] + \left(\frac{k_z}{k}\right)^2 \frac{1}{\left(\frac{\omega}{\omega_c} + n\right)} J_n^2(b) \right\} \tag{48}$$

where $b = k_\perp V/\omega_c$.

The distribution function given by equation (47) has been studied by MALMFORS (1950), by GROSS (1951), and by SEN (1952). These authors considered only the case $k_z = 0$, and also neglected the motion of the ions. MALMFORS concluded that instabilities existed. However, an error was found in his work by GROSS. GROSS obtained our equation (48) with $k_z = 0$. From this he concluded that there was no instability for $b \ll 1$. He conjectured that there was no instability for any value of b. SEN found the same dispersion relation as GROSS and was able to show numerically that instabilities exist for large enough values of b. The writer (HARRIS, 1959a and 1959c) has analysed the same problem and by use of the Nyquist criterion has been able to show that when $k_z = 0$ instabilities exist for $b > 1{\cdot}84$ (the first maximum of $J_1(b)$). Our analysis indicates that the greater instability occurs when neither $k_\perp$ nor k_z is zero. It is possible to give a sufficient condition for instability. If the motion of the ions is neglected this condition is $\omega_{pe} > \omega_{ce}$. If motion of the ions is taken into account the condition for instability becomes $\omega_{pe} > \omega_{ci}$.

We shall now examine the distribution function given by equation (39). We have already remarked that

* This interesting observation was made by ROSENBLUTH in the discussion following a paper by the writer at the Gatlinburg Conference on *Theoretical Aspects of Controlled Fusion Research*, April 27–28, 1959. The proceedings of this meeting will be published as ORNL-2805.

if $\alpha_\perp = \alpha_z$ the distribution is just the Maxwell-Boltzmann distribution for which BERNSTEIN has shown there are no instabilities. For simplicity we will limit ourselves here to the case $\alpha_z = 0$, since we know qualitatively the effect of a non-zero α_z. A thermal spread of velocities along **B** causes LANDAU damping which reduces the growth rate of the instability. Substituting equation (39) into equation (46), performing the integrations and letting α_z become zero gives

$$1 = \sum \frac{\omega_p^2}{\omega_c^2} \sum_{n=-\infty}^{+\infty} e^{-s^2/2} I_n\left(\frac{s^2}{2}\right) \times \left\{ -\frac{2n}{s^2}\left(\frac{k_\perp}{k}\right)^2 \frac{1}{\left(\frac{\omega}{\omega_c} + n\right)} + \left(\frac{k_z}{k}\right)^2 \frac{1}{\left(\frac{\omega}{\omega_c} + n\right)^2} \right\} \tag{49}$$

where $s = k_\perp \alpha_\perp / \omega_c$ and $I_n = I_{-n}$ is the Bessel function of the first kind of imaginary argument. It has been shown (HARRIS, 1959c) that there can be no instability for either $k_\perp = 0$ or $k_z = 0$; that is, for waves propagating either parallel to or perpendicular to the field. Instabilities can only occur when neither $k_\perp$ nor k_z is zero. As before a sufficient criterion for instability is found to be $\omega_{pe} > \omega_{ci}$.

The instabilities which we have just discussed may, as MALMFORS (1950) suggests, account for the noise in trochotrons. In these tubes the ions play no part and it seems reasonable to take the condition for instability to be $\omega_{pe} > \omega_{ce}$. This is also the condition found by MALMFORS. The experimental evidence of ALFVÉN *et al.* seems to support this criterion, but the data is not sufficient for a quantitative comparison of theory and experiment. It should be mentioned that similar instabilities have been found in other electron devices involving crossed electric and magnetic fields and quite different theories proposed to account for them. References to these will be found in the thesis of MILLER (1958).

4. DISCUSSION

Our examination of the dispersion relation has shown a wide variety of ways in which a plasma in a magnetic field can misbehave. We suspect that there may be other instabilities which have not been disclosed by our rather superficial examination. In particular, we feel that there is need for a more careful study of waves propagating at an angle to the magnetic field without making the approximations of the last section. Some progress along these lines has been reported by VEDFNOV and SAGDEEV (to be published). These authors assume long wavelengths and low frequencies. They assume a distribution function of the form of equation (39) and find instabilities when $\alpha_\perp > \alpha_z$ and also when $\alpha_z > \alpha_\perp$. Very few details of the calculations are given in their brief paper.

Acknowledgement—The author wishes to express appreciation for the many helpful and stimulating discussions with DR. ALBERT SIMON of the Oak Ridge National Laboratory.

APPENDIX

We shall list here some of the mathematical expressions which we considered too lengthy for inclusion in the main body of the paper.

The quantities a_{ij} are listed below:

$$a_{00} = i\sum \frac{\omega_p^2}{\omega_c} \int \left[k_\perp \frac{\partial f_0}{\partial v_\perp} I_1 + k_z \frac{\partial f_0}{\partial v_z} I_2 \right] d^3v \tag{A-1}$$

$$a_{01} = \frac{i}{c} \sum \frac{\omega_p^2}{\omega_c} \int \left[(\omega + k_z v_z) \frac{\partial f_0}{\partial v_\perp} - k_z v_\perp \frac{\partial f_0}{\partial v_z} \right] I_1 \, d^3v \tag{A-2}$$

$$a_{02} = \frac{i}{c} \sum \frac{\omega_p^2}{\omega_c} \int \left[(\omega + k_z v_z) \frac{\partial f_0}{\partial v_\perp} - k_z v_\perp \frac{\partial f_0}{\partial v_z} \right] I_3 \, d^3v \tag{A-3}$$

$$a_{03} = \frac{i}{c} \sum \frac{\omega_p^2}{\omega_c} \int \left[\omega \frac{\partial f_0}{\partial v_z} I_2 + k_\perp \times \left(v_\perp \frac{\partial f_0}{\partial v_z} - v_z \frac{\partial f_0}{\partial v_\perp} \right) I_1 \right] d^3v \tag{A-4}$$

The values of a_{10}, a_{11}, a_{12} and a_{13} can be found from the above equations by including a factor $(v_\perp/c) \cos\phi$ in the integrand. Thus

$$a_{10} = \frac{i}{c} \sum \frac{\omega_p^2}{\omega_c} \int \left[k_\perp \frac{\partial f_0}{\partial v_\perp} I_1 + k_z \frac{\partial f_0}{\partial v_z} I_2 \right] v_\perp \cos\phi \, d^3v \tag{A-5}$$

etc. The values of a_{20}, a_{21}, a_{22} and a_{23} are found by including the factor $(v_\perp/c) \sin\phi$ in the integrand of the expressions for a_{00}, a_{01}, a_{02}, a_{03}. The values of a_{30}, a_{31}, a_{32} and a_{33} are found by including the factor (v_z/c) in the integrand of the expressions for a_{00}, a_{01}, a_{02}, a_{03}. Thus

$$a_{20} = \frac{i}{c} \sum \frac{\omega_p^2}{\omega_c} \int \left[k_\perp \frac{\partial f_0}{\partial v_\perp} I_1 + k_z \frac{\partial f_0}{\partial v_z} I_2 \right] v_\perp \sin\phi \, d^3v \tag{A-6}$$

$$a_{30} = \frac{i}{c} \sum \frac{\omega_p^2}{\omega_c} \int \left[k_\perp \frac{\partial f_0}{\partial v_\perp} I_1 + k_z \frac{\partial f_0}{\partial v_z} I_2 \right] v_z \, d^3v \tag{A-7}$$

etc.

It will be seen from the above expressions that the

dependence of the integrands upon the angle ϕ occurs only in I_1, I_2, I_3, $\cos\phi$ and $\sin\phi$ and that the integration over ϕ can be performed without specifying f_0. We shall give below the necessary integrals.

$$\int_0^{2\pi} I_1\, d\phi = 2\pi \sum_{n=-\infty}^{+\infty} \frac{nJ_n^2(b)}{b(a+in)} \tag{A-8}$$

$$\int_0^{2\pi} I_1 \cos\phi\, d\phi = 2\pi \sum_{n=-\infty}^{+\infty} \frac{n^2 J_n^2(b)}{b^2(a+in)} \tag{A-9}$$

$$\int_0^{2\pi} I_1 \sin\phi\, d\phi = -\frac{\pi}{i} \sum_{n=-\infty}^{+\infty} \frac{n}{b(a+in)} \frac{d}{db} J_n^2(b) \tag{A-10}$$

$$\int_0^{2\pi} I_2\, d\phi = 2\pi \sum_{n=-\infty}^{+\infty} \frac{J_n^2(b)}{(a+in)} \tag{A-11}$$

$$\int_0^{2\pi} I_2 \cos\phi\, d\phi = \int_0^{2\pi} I_1\, d\phi \tag{A-12}$$

$$\int_0^{2\pi} I_2 \sin\phi\, d\phi = \frac{\pi}{i} \sum_{n=-\infty}^{+\infty} \frac{1}{(a+in)} \frac{d}{db} J_n^2(b) \tag{A-13}$$

$$\int_0^{2\pi} I_3\, d\phi = \int_0^{2\pi} I_2 \sin\phi\, d\phi \tag{A-14}$$

$$\int_0^{2\pi} I_3 \cos\phi\, d\phi = -\int_0^{2\pi} I_1 \sin\phi\, d\phi \tag{A-15}$$

$$\int_0^{2\pi} I_3 \sin\phi\, d\phi = 2\pi \sum_{n=-\infty}^{+\infty} \frac{1}{(a+in)} \left[\frac{d}{db} J_n(b)\right]^2 \tag{A-16}$$

In the above equations J_n is the Bessel function of first kind and of order n,

$$a = i\left(\frac{\omega + k_z v_z}{\omega_c}\right), \tag{A-17}$$

$$b = \frac{k_\perp v_\perp}{\omega_c}. \tag{A-18}$$

In deriving these equations we have made use of the identity

$$e^{ib\sin\phi} = \sum_{n=-\infty}^{+\infty} J_\epsilon(b) e^{in\phi} \tag{A-19}$$

and several well-known identities involving Bessel functions. The calculations are straightforward but tedious.

When f_0 has the form of the Maxwell-Boltzmann distribution function a formula given by WATSON (1945) is useful in performing the integration over $v_\perp$. This formula is

$$\int_0^\infty \rho\, d\rho e^{-\rho^2} J_n(s_1\rho) J_n(s_2\rho) = \frac{1}{2} e^{-(s_1^2+s_2^2/4)} I_n\left(\frac{s_1 s_2}{2}\right) \tag{A-20}$$

where I_n is Bessel function of the first kind of imaginary argument. From this formula may be derived the three integrals necessary for the v_1 integration. These are

$$\int_0^\infty \rho\, d\rho e^{-\rho^2} J_n^2(s\rho) = \frac{1}{2} e^{s^2/2} I_n \frac{s^2}{(2)}, \tag{A-21}$$

$$\int_0^\infty \rho^2\, d\rho e^{-\rho^2} \frac{d}{d\rho} J_n^2(s\rho) = \frac{s^2}{2} e^{-s^2/2}\left[I_n' - I_n\right], \tag{A-22}$$

$$\int_0^\infty \rho^3 d\rho e^{-\rho^2} \left[\frac{d}{d\rho} J_n(s\rho)\right]^2 = \frac{s^4}{8} e^{-s^2/2} \times \left[\left(\frac{4n^2}{s^4} + 2\right) I_n - I_n'\right] \tag{A-23}$$

In the above the prime denotes a derivative with respect to the argument. The argument of I_n and I_n' is $s^2/2$.

When the distribution in v_z is Gaussian, the integrations over v_z can be expressed in terms of the integrals

$$L_1(t_n) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-z^2} \frac{dz}{(z+t_n)}, \tag{A-24}$$

$$L_2(t_n) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-z^2} \frac{z^2\, dz}{(z+t_n)}, \tag{A-25}$$

$$L_3(t_n) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-z^2} \frac{z^2\, dz}{(z+t_n)}, \tag{A-26}$$

where

$$t_n = \frac{\omega + n_c}{k_z \alpha_z}. \tag{A-27}$$

For the purposes of this paper we need only the limiting form of these integrals for small α_z or equivalently large t_n. It is easily shown that as $t_n \to \infty$,

$$L_1 \to \frac{1}{t_n}, \tag{A-28}$$

$$L_2 \to -\frac{1}{t_n^2}, \tag{A-29}$$

$$L_3 \to \frac{1}{t_n}. \tag{A-30}$$

Integrals of the form of L_1 and L_2 have been discussed by BHATNAGAR, GROSS and KROOK (1954) and by SUMI (1958). L_3 can be expressed in terms of L_i by noting that

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\beta z^2} \frac{dz}{(z+t_n)} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-z^2} \frac{dz}{(z+\sqrt{\beta}\, t_n)} = L_1(\sqrt{\beta}\, t_n). \tag{A-31}$$

Differentiation with respect to β gives

$$L_3(t_n) = -2\left[\frac{d}{d\beta} L_1(\sqrt{\beta}\, t_n)\right]_{\beta=1} \qquad \text{(A-32)}$$

REFERENCES

Alfvén H., Lindberg L., Malmfors K. G., Wallmark T. and Astrom E. (1948) *Kgl. Tek. Högsk. Handl.* NR **22**.

Bernstein I. B. (1958) *Phys. Rev.* **109**, 10.

Bernstein W., Chen F. F., Heald M. A. and Kranz A. Z. (1958) *Phys. Fluids* **1**, 430.

Bhatnagar P. L., Gross E. P. and Krook M. (1954) *Phys. Rev.* **94**, 511.

Buneman O. (1958) *Phys. Rev. (Letters)* **1**, 8.

Colgate S. A. and Furth H. P. (1958) *Science* **128**, 337.

Coor T., Cunningham S. P., Ellis R. A., Heald M. A. and Kranz A. Z. (1958) *Phys. Fluids* **1**, 411.

Dawson J. and Bernstein I. B. (1958) Paper presented at the Controlled Thermonuclear Conference, Washington, D.C., TID-7558, 360.

Fried B. D. (1959) *Phys. Fluids* **2**, 337.

Gross E. P. (1951) *Phys. Rev.* **82**, 232.

Harris E. G. (1959a) *Phys. Rev. (Letters)* **2**, 34.

Harris E. G. (1959b) *Bull. Amer. Phys. Soc., Series II* **4**, 388.

Harris E. G. (1959c) *Unstable Plasma Oscillations in a Magnetic Field.* ORNL-2728.

Kahn F. D. (1957) *J. Fluid Mech.* **2**, 610.

Kahn F. D. (1958) *Rev. Mod. Phys.* **30**, 1069.

Landau L. D. (1946) *J. Phys. U.S.S.R.* **10**, 25.

Malmfors K. G. (1950) *Ark. Fys.* **1**, 569.

Miller M. H. (1958) *Study of High Temperature Electrons Originating in Streams Flowing in Crossed d.c. Electric and Magnetic Fields.* Technical Report 26, Electron Tube Laboratory, University of Michigan, Ann Arbor, Michigan.

Rosenbluth M. (1959) *Bull. Amer. Phys. Soc., Series II* **4**, 197.

Sen H. K. (1952) *Phys. Rev.* **88**, 816.

Sumi M. (1958) *J. Phys. Soc., Japan* **13**, 1476.

Vedenov A. A. and Sagdeev R. Z., in *Plasma Physics and the Problem of Controlled Thermonuclear Reactions.* Pergamon Press Vol. III. To be published.

Watson G. N. (1945) *Theory of Bessel Functions*, p. 395. Cambridge University Press, Cambridge, Mass.

Weibel E. S. (1959) *Phys. Rev. (Letters)* **2**, 83.

VOL.
2
961

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 146 to 153. Pergamon Press Ltd. Printed in Northern Ireland

A QUASI-LINEAR MODEL OF PLASMA SHOCK STRUCTURE IN A LONGITUDINAL MAGNETIC FIELD*

E. N. PARKER
Enrico Fermi Institute for Nuclear Studies and Department of Physics, The University of Chicago
Chicago 37, Illinois

Abstract—It is pointed out that interpenetration of two plasma streams along a uniform magnetic field is prevented by the hose instability. Thus the shock thickness resulting from supersonic streaming of a tenuous plasma along a uniform magnetic field may be very much less than the collision length. We suggest that an actual stationary shock front under these conditions is formed by the hose interaction between the incoming supersonic stream and a precursor, consisting of ions evaporating forward from the non-linear disordering which results when the hose instability reaches large amplitude. Treating the growth of the hose instability with the WKB approximation applied to the linearized equations, we show that the scale of the shock front is characterized by the ion and electron Larmor radii, and that the density in the precursor relaxes to a small but non-vanishing value, proportional to one over the square of the Mach number, infinitely far ahead of the shock.

1. INTRODUCTION

ONE of the major obstacles in our path toward a theoretical understanding of the particle acceleration and electromagnetic wave emission from violent astrophysical phenomena such as the solar flare, the nova and supernova, the Crab Nebula, etc. is the lack of a quantitative theoretical model of shock structure in a tenuous ionized gas. Laboratory data are difficult to obtain primarily because of scaling difficulties and it has not yet been possible to solve the Boltzmann equation with a sufficient degree of non-linear detail to deduce what shock structure might be expected.

The purpose of this paper is to explore the possibility that a shock front in a tenuous plasma moving along a uniform magnetic field can be understood at least qualitatively on the basis of two principles viz. the growth of linear instabilities in supersonic interpenetrating streams, and the production of disordering and 'entropy' when the instabilities reach large amplitude.

There are at least three known dynamical mechanisms which appear capable of producing shock fronts in the absence of collisions, and they are (1) magnetic field perpendicular to the shock velocity, (2) magnetic field parallel to the shock velocity, and (3) co-operative electrostatic plasma interaction. The first case was pointed out some years ago, with the suggestion that the resulting shock thickness would be comparable to the electron (COLGATE, 1957) or ion (PETSCHEK, 1958) Larmor radius. Direct solution of the dynamical equations for a cold plasma has subsequently shown (LONGMIRE, 1957; GARDNER, GOERTZEL, GRAD, MORAWETZ, ROSE and RUBIN, 1959) that the scale of the shock is related to the harmonic mean of the electron and ion Larmor radii. Unfortunately the analytic solutions that have been given are necessarily reversible, so that they are unable to treat in a quantitative way the 'entropy' production associated with a true shock phenomenon. It has been argued plausibly and demonstrated by numerical calculation that, as a consequence of instability, non-linear breaking and phase mixing, there should at least be macroscopic disordering produced in the oscillations which occur behind the shock front (LONGMIRE, 1957; GARDNER, GOERTZEL, GRAD, MORAWETZ, ROSE and RUBIN, 1959; BUNEMANN, 1959, DAWSON, 1959). Thus one can fix a scale, perhaps comparable to the ion Larmor radius over which the plasma motion may be regarded as disordered. To the macroscopic observer, such disordering is effectively true entropy.

We have previously discussed (PARKER, 1958 and 1959) case (3), pointing out that under many astrophysical circumstances it may be the strongest of the three interactions. Unfortunately the co-operative electrostatic interaction depends quantitatively rather critically on both the form of the ion velocity distribution and the electron temperature, so that the construction of a quantitative model requires the making of a number of explicit assumptions. Thus, before tackling this more difficult problem we have felt it advisable to work out the relatively simpler case (2), in which the magnetic field is parallel to the gas velocity.

We shall follow the lead of the previous work on case (1) to the extent that we shall suppose the 'entropy' to be generated as the result of the breaking and

* This research is supported by the Geophysical Research Directorate, Air Force Cambridge Research Center, Air Research and Development Command, Contract AF 19 (604)-4554.

disordering of large-amplitude non-linear oscillations produced in the front. We shall make the usual assumption that a stationary shock structure exists. We shall regard a shock front to be the region of overtaking collision in which a supersonic stream runs into the rear of a hotter and more dense subsonic stream.

Up to the point where the non-linear effects produce disorder, we shall suppose that the linear theory is at least qualitatively correct. The resulting shock structure will be a model, then, based upon the two qualitative fragments of information, linear growth and non-linear breaking, currently available. As we shall see, a long precursor is a basic result of the physical picture presented in this paper. The precursor is essentially the evaporation of ions from the region of disordering and its existence or non-existence probably forms the simplest experimental test of the model.

2. INTERPENETRATING STREAM INTERACTION

Consider the formation of a shock front in the presence of a magnetic field parallel to the plasma motion. Our basic point of view will be the following argument: Suppose that there were no shock interaction in the absence of collisions when a supersonic stream overtakes a subsonic stream flowing along a uniform magnetic field. Then obviously the supersonic stream would flow freely into the slower stream until finally we would have two infinitely long interpenetrating streams. But we shall demonstrate that two plasma streams interpenetrating along a magnetic field will interact strongly, so that they never could have interpenetrated very far in the first place. In this way we shall demonstrate that a magnetic field parallel to the motion can produce the equivalent of a collision between two streams, so that the formation of a shock front of some sort must result when the relative stream velocities are supersonic.

Thus consider a stream of plasma of density N_1 ions and electrons per unit volume flowing with velocity U in the positive z-direction along the uniform magnetic field $\mathbf{e}_z B$, through a stationary plasma cloud of density N_2. It is immediately obvious that the interpenetration is subject to the hose instability. Let $\mathbf{e}_x u_1(z, t)$ and $\mathbf{e}_x u_2(z, t)$ represent plane transverse velocity perturbations in the streaming and stationary plasma respectively. Let $\mathbf{e}_x b(z, t)$ denote the corresponding magnetic perturbation. Then since both plasmas are bound to the magnetic field according to the usual hydromagnetic equation

$$\partial \mathbf{B}/\partial t = \nabla \times (\mathbf{v} \times \mathbf{B})$$

we have

$$\partial b/\partial t = B_0\, \partial u_1/\partial z - U\, \partial b/\partial z, \tag{1}$$

$$\partial b/\partial t = B_0\, \partial u_2/\partial z. \tag{2}$$

The Lorentz force $(\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi$ is exerted by the field onto both plasmas, so that the linearized equation of motion is

$$N_1 M(\partial u_1/\partial t + U\, \partial u_1/\partial z) + N_2 M\, \partial u_2/\partial t = +(B_0/4\pi)\, \partial b/\partial z. \tag{3}$$

Assuming solutions of the form $\exp i(\omega t + kz)$ with amplitudes UA_1, UA_2, $B_0 A_3$ for u_1, u_2 and b respectively, we readily find the amplitude relations

$$A_1 = A_3(1 + \omega/kU), \qquad A_2 = A_3\omega/kU \tag{4}$$

and the dispersion relation

$$B_0 k\{N_1 M(\omega + kU)^2 + N_2 M\omega^2 - k^2 B_0^2/4\pi\} = 0 \tag{5}$$

In terms of the net hydromagnetic velocity

$$C^2 = B_0^2/4\pi(N_1 + N_2)M$$

the dispersion relation has the root

$$i\omega = kU\{-iN_1/(N_1 + N_2) + [N_1 N_2/(N_1 + N_2)^2 - C^2/U^2]^{\frac{1}{2}}\} \tag{5a}$$

which is unstable provided only that the stream velocity sufficiently exceeds the hydromagnetic velocity,

$$U > C(N_1 + N_2)/(N_1 N_2)^{\frac{1}{2}}. \tag{6}$$

The perturbation is fixed in the frame of reference of the centre of mass of the two plasmas; the characteristic time of growth, and slowing of the relative stream velocity, is of the order of $1/kU$.

We conclude, then, that one plasma stream cannot freely penetrate another in the presence of a longitudinal magnetic field. The hose instability provides a strong interaction between the streams, so that they 'collide' instead of interpenetrating. The scale characteristic of the collision region is given by the real part of ik when ω is taken to be real,

$$ik = \omega\{-iN_1/(N_1 + N_2) + [N_1 N_2/(N_1 + N_2)^2 - C^2/U^2]^{\frac{1}{2}}\} \times U/[U^2 N_1/(N_1 + N_2) - C^2]. \tag{7}$$

Since a plasma does not respond to a magnetic interaction faster than the cyclotron frequency of its ions and electrons, we can place an upper limit on ω probably somewhere between eB/Mc and eB/mc. Thus ik is characterized by the ion and electron Larmor radii and the characteristic shock thickness with a longitudinal magnetic field is of the same order of magnitude as with a transverse field.

Stream deceleration

Having demonstrated that two plasma streams interpenetrating along a uniform magnetic field will develop an instability, let us now calculate how rapidly the relative motion of the two streams will be decelerated. Since we will need the deceleration in the next section where we work out an explicit model for the resulting shock structure, we must consider the deceleration of a uniform stream of density N_1, flowing into a cloud of increasing density $N_2(z)$ along the field B_0.

We eliminate u_1 and b from (1), (2) and (3) by differentiating (3) with respect to z and t. We then use (1) to eliminate u_1 and (2) to eliminate b, obtaining finally

$$N_1\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial z}\right)^2\frac{\partial u_2}{\partial z}+N_2(z)\frac{\partial^3 u_2}{\partial t^2\,\partial z}-\frac{B_0^2}{4\pi M}\frac{\partial^3 u_2}{\partial z^3}=-\frac{dN_2}{dz}\frac{\partial^2 u_2}{\partial t^2}$$

we let

$$u_2(z,t)=\psi(z)\exp i\omega t$$

so that

$$\left[N_1\left(i\omega+U\frac{d}{dz}\right)^2-\omega_2 N_2(z)\right]\frac{d\psi}{dz}-\frac{B_0^2}{4\pi M}\frac{d^3\psi}{dz^3}=+\omega^2\frac{dN_2}{dz}\psi.$$

Assuming that $N_2(z)$ is a slowly varying function of z, we apply the WKB approximation with

$$\psi=\exp\left(S_0(z)+S_1(z)+\ldots\right).$$

Then the zero-order terms yield

$$\frac{dS_0}{dz}=\omega\left\{\frac{-iU+[N_2U^2/N_1-C_1^2(1+(N_2/N_1)\,N_2/N_1)]}{U^2-C_1^2}\right\} \tag{8}$$

where $C_1^2\equiv B_0^2/4\pi N_1 M$. The first-order terms yield

$$\frac{dS_1}{dz}+\frac{1}{2Q}\frac{dQ}{dz}=\frac{\omega^2}{ZN_1Q}\frac{dN_2}{dz} \tag{9}$$

where

$$Q=3(U^2-C_1^2)(dS_0/dx)^2+4i\omega U\,dS_0/dx-\omega^2-\omega^2N_2(z)/N_1$$

$$=\frac{2\omega^2}{(U^2-C_1^2)}V(z)[V(z)-iU]$$

where

$$V(z)=\left[\frac{N_2}{N_1}(U^2-C_1^2)-C_1^2\right]^{1/2}.$$

Hence, upon integration of (9) we have

$$S_1(z)=-\tfrac{1}{2}\ln Q+\tfrac{1}{2}\ln[V(z)-iU]$$
$$=-\tfrac{1}{2}\ln V(z)-\tfrac{1}{2}\ln[2\omega^2/(U^2-C_1^2)]. \tag{10}$$

Thus we have

$$u_2(z,t)=u_2(z_0,0)[V(z_0)/V(z)]^{\frac{1}{2}}\times\exp\left[i\omega\left(t-\frac{Uz}{U^2-C_1^2}\right)\right]\times\exp\left[\frac{\omega}{U^2-C_1^2}\int_{z_0}^{z}d\lambda V(\lambda)\right]. \tag{11}$$

From (2) it follows that

$$b(z,t)=-B_0\frac{u_2(z_0,0)(iV(z)+U)}{U^2-C_1^2}\left[\frac{V(z_0)}{V(z)}\right]^{1/2}\times\exp\left[i\omega\left(t-\frac{Uz}{U^2-C_1^2}\right)\right]\times\exp\left[\frac{\omega}{U^2-C_1^2}\int_{z_0}^{z}d\lambda V(\lambda)\right], \tag{12}$$

and from (1)

$$u_1(z,t)=-u_2(z_0,0)\left(\frac{C_1^2+iUV}{U^2-C_1^2}\right)\left[\frac{V(z_0)}{V(z)}\right]^{\frac{1}{2}}\times\exp\left[i\omega\left(t-\frac{Uz}{U^2-C_1^2}\right)\right]\times\exp\left[\frac{\omega}{U^2-C_1^2}\int_{z_0}^{z}d\lambda\,V(\lambda)\right]. \tag{13}$$

Note that the group velocity and phase velocity are both equal to $(U^2-C_1^2)/U$.

Now it follows that

$$\langle u_1^2(z,t)\rangle=\tfrac{1}{2}u_2^2(z_0,0)\frac{V(z_0)}{V(z)}\left[\frac{U^2N_2(z)/N_1-C_1^2}{U^2-C_1^2}\right]\times\exp\left[\frac{2\omega}{U^2-C_1^2}\int_{z_0}^{z}d\lambda V(\lambda)\right], \tag{14}$$

and

$$\langle u_2^2(z,t)\rangle=\tfrac{1}{2}u_2^2(z_0,0)\frac{V(z_0)}{V(z)}\times\exp\left[\frac{2\omega}{U^2-C_1^2}\int_{z_0}^{z}d\lambda V(\lambda)\right]. \tag{15}$$

The energy flux in the stationary plasma is

$$F_2=\tfrac{1}{2}N_2M\langle u_2^2(z,t)\rangle(U^2-C_1^2)/U$$

so that, to the order of approximation we are using, the energy input per unit volume and time is

$$\frac{dF_2}{dz}=\frac{1}{2}Mu_2^2(z_0,0)\frac{V(z_0)}{U}\omega N_2(z)\times\exp\left\{\frac{2\omega}{U^2-C_1^2}\int_{z_0}^{z}d\lambda V(\lambda)\right\}.$$

VOL 2 196

For the moving plasma we have an energy flux

$$F_1 = \tfrac{1}{2}N_1M\langle u_1^2(z, t)\rangle(U^2 - C_1^2)/U$$

and an energy input

$$\frac{dF_1}{dz} = \frac{1}{2}Mu_2^2(z_0, 0)\frac{V(z_0)}{U}\omega N_1\left[\frac{U^2N_2(z)/N_1 - C_1^2}{U^2 - C_1^2}\right] \times \exp\left\{\frac{2\omega}{U^2 - C_1^2}\int_{z_0}^{z} d\lambda V(\lambda)\right\}.$$

Since the energy input to the waves comes from the relative kinetic energy of the two plasmas, we have that the retarding force per unit volume must be f, where

$$Uf = dF_1/dz + dF_2/dz,$$

or

$$f = \tfrac{1}{2}\omega MV(z_0)[u_2^2(z_0, 0)/U^2] \times \left[\frac{N_2(z)(2U^2 - C_1^2) - C_1^2N_1^2}{U^2 - C_1^2}\right] \times \exp\left[\frac{2\omega}{U^2 - C_1^2}\int_{z_0}^{z} d\lambda V(\lambda)\right]. \quad (16)$$

The stream deceleration is just

$$\frac{dU}{dt} = \frac{f(z, t)}{N_1M}. \quad (17)$$

3. SHOCK STRUCTURE

A. *Basic equations*

Suppose that a stream of supersonic plasma of uniform density N_1 flows along a magnetic field B_0 with velocity U from $z = -\infty$ into the rear of a subsonic stream in the vicinity of $z = 0$. As a result of the linear hose instability the interpenetration of the two streams generates growing transverse waves. The waves soon reach large amplitude, so that at some distance back into the subsonic stream it is not unreasonable to assume that the growing non-linearity of the waves results in a region of breaking and disorder. Thus we postulate *two distinct* regions, one of linear growth and the other of non-linear disordering. This postulate implies, first, that the instability begins to grow from small random fluctuations ahead of the shock, rather than from some sort of disturbance propagated with a velocity in excess of U from the disordered region and, second, that ions are not transferred in significant numbers from one stream to the other until a final break-up occurs in the non-linear region about $z = 0$. It is these two assumptions which permit the quasi-linear treatment developed below.

Behind the region where the disordering is generated we have the shocked plasma, which we take to be of mean density N_3, effective temperature T_3, and flow velocity U_3. And because there has been disordering in the vicinity of $z = 0$, we would presume that the Rankine–Hugoniot relations should relate N_3 and T_3 far behind the shock to N_1 and T_1 far ahead.

Ahead of the region of active disordering, i.e. in $z < 0$, the transverse waves are growing according to the linear theory, as a result of the interaction between the incoming supersonic stream of density N_1 and the shocked plasma with density $N_2(z)$ and temperature $T_2(z)$ of the subsonic stream. We shall refer to the shocked plasma ahead of the region of disordering as the *precursor*. In a 'stationary' shock, such as we shall consider, with the breaking largely confined to a region behind the precursor, there will be no net flow of plasma in the precursor. If F is the total momentum transfer rate between the supersonic and subsonic streams in the region of disorder, then F plus the gas pressure in the precursor at the forward side of the disordering region, $z = 0$, must equal the gas pressure far behind,

$$2N_2(0)kT_2(0) + F = 2N_3kT_3. \quad (18)$$

We have no way of calculating F, but our assumption that no disordering occurs until the waves reach large amplitude implies that $2N_2(0)kT_2(0)$ is a significant fraction of $2N_3kT_3$, i.e. a sizeable portion of the momentum is transferred in the region of linear wave growth.

The momentum transfer from the supersonic stream to the precursor is given by (16), so that for a stationary precursor we must have

$$f(z) = dp_2/dz, \quad (19)$$

which becomes an integro-differential equation for $p_2(z)$ in the precursor. We suppose that $T_2(z)$ is related to $N_2(z)$ in the simplest possible way,

$$T_2(z) = T_2(0)[N_2(z)/N_2(0)]^{q-1} \quad (20)$$

where $q \geq 1$. Thus (19) becomes

$$q\phi^{q-1}\,d\phi/dz = KV(z)_0(\phi - \alpha) \times \exp\left\{\gamma\int_{z_0}^{z} d\lambda[\phi(\lambda) - \beta]^{\frac{1}{2}}\right\} \quad (21)$$

where, with $s = C_1/U^{z_0}$, we have

$$K = \frac{\omega Mu_2^2(z_0, 0)(2 - s^2)}{4kT_2(0)U^2(1 - s^2),}, \quad (22)$$

$$\alpha = N_1s^2/N_2(0)(2 - s^2), \quad (23)$$

$$\beta = N_1s^2/N_2(0)(1 - s^2), \quad (24)$$

$$\gamma = 2\omega N_2^{\frac{1}{2}}(0)/UN_1^{\frac{1}{2}}(1 - s^2)^{\frac{1}{2}} = 2\omega\beta^{\frac{1}{2}}/sU, \quad (25)$$

and the dependent variable is

$$\phi(z) = N_2(z)/N_2(0). \tag{26}$$

We divide both sides of (21) by $\phi - \alpha$, differentiate with respect to z, and use (21) to eliminate the integral. In terms of the variable

$$\xi = \gamma z$$

we obtain

$$\frac{d}{d\xi}\left(\frac{\phi^{q-1}}{\phi - \alpha}\frac{d\phi}{d\xi}\right) = \frac{(\phi - \beta)^{\frac{1}{2}}\phi^{q-1}}{\phi - \alpha}\frac{d\phi}{d\xi}. \tag{27}$$

The solution of this differential equation is easily reducible to quadrature for integral and half integral values of q. We will obtain two constants of the integration. One is consumed in making $\phi(0) = 1$. The other is needed to make ϕ, the solution of a second-order differential equation, the solution of the integro-differential equation from which the second-order equation was derived. This we achieve by adjusting the constant so that (21) is satisfied algebraically at $z = z_0$. We shall presume that $\phi(-\infty)$ is small, if not identically zero.

Integrating (27) once we obtain

$$\frac{d\phi}{d\xi} = [A_q + Q_q(\phi)](\phi - \alpha)\phi^{1-q},$$

where

$$Q_q(\phi) = \int \frac{d\phi\ \phi^{q-1}(\phi - \beta)^{\frac{1}{2}}}{\phi - \alpha}. \tag{28}$$

In order that $\phi(\xi)$ satisfy (21) at $z = z_0$, where ϕ has a value ϕ_1, we must have

$$A_q = \frac{1}{q}\xi(1 - s^2)^{\frac{1}{2}}(\phi_1 - \beta)^{\frac{1}{2}} - Q_q(\phi_1) \tag{29}$$

where

$$\begin{aligned}\zeta &\equiv KU[N_2(0)/N_1]^{\frac{1}{2}}/\gamma \\ &= Mu_2^2(z_0, 0)/4kT_2(0) \\ &\ll 1.\end{aligned} \tag{30}$$

Thus the solution of (27) may be written

$$\xi = -\int_{\phi}^{1} \frac{d\phi\ \phi^{q-1}/(\phi - \alpha)}{\{Q_q(\phi) - Q_q(\phi_1) + (\zeta/q)(1 - s^2)^{\frac{1}{2}}(\phi_1 - \beta)^{\frac{1}{2}}\}} \tag{31}$$

The limits of integration have been chosen so that $\phi(0) = 1$.

In the limit of infinite Mach number, we have $s \to 0$ and $\beta = 2\alpha \to 0$. Thus for all $q > \frac{1}{2}$

$$Q_q(\phi) = \phi^{q-\frac{1}{2}}/(q - \tfrac{1}{2}), \tag{32}$$

$$A_q = \phi_1^{\frac{1}{2}}[\zeta/q - \phi_1^{q-1}/(q - \tfrac{1}{2})]. \tag{33}$$

When $s \neq 0$, $Q_q(\phi)$ is readily evaluated for integral and half integral values of q. We are not sure what value of q to expect in an actual shock front. It is clear that q depends upon the form of the ion velocity distribution generated in the region of non-linear break-up. Qualitatively we would expect the distribution to be not unlike the Maxwellian form up to moderate ion velocities, but lacking the high velocity tail normally achieved in complete statistical equilibrium. If we could assume that the interaction force $f(z)$ in the precursor acted equally on all particles, so that the stationary Boltzmann equation for the ion velocity distribution function $\psi(v, z)$ could be written

$$mv\ d\psi/dz + f(z)\ d\psi/dv = 0,$$

then it is readily shown that a Maxwellian distribution yields an isothermal precursor, $q = 1$, whereas cutting off the high velocity tails to give a simple parabolic distribution yields the adiabatic relation for three degrees of freedom, $q = 5/3$. A square distribution yields $q = 3$, corresponding to the adiabatic relation for one degree of freedom. Thus we expect that q will be somewhat greater than one, and consequently we will discuss the special values $q = 1$, 3/2, 2, 3. We trust that these values of q will adequately cover the necessary range.

$$Q_1(\phi) = 2\{(\phi - \beta)^{\frac{1}{2}} - (\beta - \alpha)^{\frac{1}{2}} \times \arctan[(\phi - \beta)/(\beta - \alpha)^{\frac{1}{2}}\} \tag{34}$$

$$\begin{aligned}Q_{3/2}(\phi) = {} & \phi^{\frac{1}{2}}(\phi - \beta)^{\frac{1}{2}} \\ & + (2\alpha - \beta)\ln\left[\frac{(\phi - \beta)^{\frac{1}{2}} + \phi^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\right] \\ & - \alpha^{\frac{1}{2}}(\beta - \alpha)^{\frac{1}{2}}\left[\arcsin\frac{\alpha^{\frac{1}{2}}\phi^{\frac{1}{2}} - \beta}{\beta^{\frac{1}{2}}(\phi^{\frac{1}{2}} - \alpha^{\frac{1}{2}})}\right. \\ & \left. + \arcsin\frac{\alpha^{\frac{1}{2}}\phi^{\frac{1}{2}} + \beta}{\beta^{\frac{1}{2}}(\phi^{\frac{1}{2}} + \alpha^{\frac{1}{2}})}\right]\end{aligned} \tag{35}$$

$$Q_2(\phi) = \tfrac{2}{3}(\phi - \beta)^{3/2} + \alpha Q_1(\phi), \tag{36}$$

$$Q_3(\phi) = \tfrac{2}{5}(\phi - \beta)^{5/2} + \tfrac{2}{3}(\alpha + \beta) \times (\phi - \beta)^{3/2} + \alpha^2 Q_1(\phi). \tag{37}$$

We shall now investigate the form of the precursor for the simple case of infinite Mach number, before going on to the somewhat more complicated circumstances occurring with finite Mach number.

B. *Infinite Mach number*

We find that the form of the precursor when $q = 1$ is distinct from the form when $q > 1$. Thus we consider first Case (1): $q = 1$. We have immediately

$$\phi(\xi) = \frac{(A_1/2)^2}{[(1 + A_1/2)\exp(-A_1\xi/2) - 1]^2} \tag{38}$$

and

$$A_1 = -\phi_1^{\frac{1}{2}}(2-\zeta).$$

Since we have postulated that $\zeta \ll 1$, we have

$$A_1 \cong -2\phi_1^{\frac{1}{2}},$$

and A_1 is negative. It follows from (38) that

$$A_1 = -2\phi^{\frac{1}{2}}(-\infty),$$

so that $\phi(-\infty)$ may be made as small as we like by making $|A_1|$ sufficiently small. To calculate the distance ξ_1 out to where $(z = z_0)$ the oscillations begin to grow, we again use (21), which requires that

$$(d\phi/d\xi)_1 = \phi_1^{3/2}\zeta.$$

From (38) we obtain

$$(d\phi/d\xi)_1 = 2\phi_1^{3/2}(1 + A_1/Z)\exp(-A_1\xi_1/2)$$

so that

$$(1 + A_1/z)\exp(-A_1\xi_1/2) = \zeta/2$$

and

$$\xi_1 \cong [2/\phi_1^{\frac{1}{2}}(-\infty)]\ln(2/\zeta). \tag{39}$$

We see that as we make $\phi(-\infty)$ small, the length ξ_1 of the precursor becomes large without bound.

Now

$$\phi(\xi) = \frac{\phi(-\infty)}{\{1 - [1 - \phi^{\frac{1}{2}}(-\infty)]\exp[\phi^{\frac{1}{2}}(-\infty)\xi]\}^2} \tag{40}$$

When $|\xi\phi^{\frac{1}{2}}(-\infty)| \ll 1$, we have

$$\phi(\xi) \cong \frac{1}{(1-\xi)^2}\left\{1 - \phi^{\frac{1}{2}}(-\infty)\frac{2\xi(1-\xi/2)}{1-\xi} + 0[\phi(-\infty)\xi^2]\right\}. \tag{41}$$

The slope at $z = 0$ is just $d\phi/dz = 2\gamma$, and the form of the precursor is independent of $\phi(-\infty)$ and ξ_1 everywhere except far out on the precursor where $\xi = 0(\xi_1) = 0[1/\phi^{\frac{1}{2}}(-\infty)]$.

Case 2: $q > 1$. We have

$$\xi = -(q - \tfrac{1}{2})\int_{\phi}^{1} \frac{d\phi\,\phi^{q-2}}{\phi^{q-\frac{1}{2}} + (q-\frac{1}{2})A_q}. \tag{42}$$

With $q > 1$ it is not necessary that A_q be negative in order to satisfy (33). Thus we may avoid ξ going to $-\infty$ while ϕ is still non-vanishing. Consider then $A_q \geq 0$. If $A_q = 0$, as a consequence of $\phi_1 = 0$ or $\phi_1 = [(2q-1)\zeta/2q]^{1/(q-1)}$, we have immediately from (42) that

$$\phi(\xi) = 1/[1 - \xi/(2q-1)]^2, \tag{43}$$

so that ϕ goes to zero as $\xi \to -\infty$. On the other hand, if $A_q > 0$, as a consequence of $0 < \phi_1 < [(2q-1)\zeta/2q]^{1/(q-1)}$, we see immediately that the precursor does not extend all the way to infinity. The precursor ends at $-\xi_2$, i.e. $\phi(-\xi_2) = 0$, where

$$\xi_2 = (q - \tfrac{1}{2})\int_0^1 \frac{d\phi\,\phi^{q-2}}{\phi^{q-\frac{1}{2}} + (q-\frac{1}{2})A_q}. \tag{44}$$

Since ζ is a very small quantity, it follows from (33) that A_q must also be very small. Thus, with

$$\chi \equiv 1/[(q-\tfrac{1}{2})A_q]^{2/(2q-1)} \gg 1, \tag{45}$$

we may write

$$\xi_2 = (q-\tfrac{1}{2})\chi^{\frac{1}{2}}\left\{\int_0^{\infty}\frac{du u^{q-2}}{1+u^{q-\frac{1}{2}}} - \int_{\chi}^{\infty}\frac{du u^{q-2}}{1+u^{q-\frac{1}{2}}}\right\} \tag{46}$$

$$\cong \chi^{\frac{1}{2}}\left\{\frac{\pi}{\sin[(q-1)\pi/(q-\frac{1}{2})]} - (q-\tfrac{1}{2})\left[\frac{2}{\chi^{\frac{1}{2}}} - \frac{1}{q\chi^q} + \frac{2}{(4q-1)\chi^{(4q-1)/2}} + \cdots\right]\right\} \tag{47}$$

in inverse powers of χ. It is immediately obvious in this case that there is a minimum precursor length, which occurs for the maximum value of A_q,

$$(A_q)_{\max} = [4(q-1)/(2q-1)](\zeta/2q)^{(2q-1)/2(q-1)}$$

for

$$\phi_1 = (\zeta/2q)^{1/(q-1)}.$$

The minimum length is

$$(\xi_2)_{\min} \cong \frac{\pi\chi_{\min}^{\frac{1}{2}}}{\sin[\pi/(2q-1)]} - (2q-1) + 0(1/\chi^{q-\frac{1}{2}}), \tag{48}$$

where

$$\chi_{\min} = \frac{2^{1/(2q-1)(q-1)}}{(q-1)^{2/(2q-1)}}\left(\frac{q}{\zeta}\right)^{1/(q-1)}.$$

Our job is now to decide the correct value of A_q on physical grounds. Note first that for any $A_q \geq 0$ we have

$$\phi(\xi) \cong 1/[1 - \xi/(2q-1)]^2 + 0(A_q/\phi^q) \tag{49}$$

for $0 > \xi \geq -(2q-1)/A_q^{\frac{1}{2}}$. Thus the form of the shock front is independent of our choice of A_q everywhere except far out on the precursor. The characteristic scale λ is just

$$1/\lambda \equiv (d\ln\phi/dz)_{z=0} = \gamma(q-\tfrac{1}{2})$$

or

$$\lambda = \left(\frac{2q-1}{4}\right)\frac{U}{\omega}\left[\frac{N_1}{N_2(0)}\right]^{\frac{1}{2}}. \tag{50}$$

Clearly, then, A_q will be determined by conditions at the leading end of the precursor.

We have defined ξ_1 to be the position from which

the random noise fluctuations begin to grow as a consequence of the hose instability. It follows, then, that ϕ must vanish everywhere to the left of ξ_1, for the simple reason that random noise fluctuations to the left of ξ_1 would grow if ϕ did not vanish there and ξ_1 would then not be their starting position. Since $\phi(\xi)$ is a continuous function, we must have $\phi_1 = 0$ in order that ϕ vanish for all $\xi < \xi_1$. It follows immediately from (33) that $A_q = 0(q > 1)$, so that the precursor is given by (43) or (49) for all values of ξ. We cannot avoid a precursor which extends asymptotically to infinity; the finite precursor, resulting when $A_q > 0$, does not satisfy the boundary conditions properly.

The characteristic scale of the steep portion of the precursor is given by (50). The total mass in the precursor is readily shown to be $(2q - 1)N_2(0)M/\gamma$ for all $q \geq 1/2$. Thus, even though the precursor is, in a sense, infinitely long, its characteristic scale and mass are both finite.

C. *Finite Mach number*

Consider the case where the Mach number U/C_1 is finite, i.e. $s > 0$. It is immediately obvious from (7) or (16) that there is no interaction between the incoming supersonic stream and the precursor beyond the zero of $V(z)$, given by $\phi(\xi) = \beta$. Thus $\phi(\xi)$ decreases from $\phi(0) = 1$ to $\phi(\xi) = \beta$ at, say, $\xi = \xi_c$, and remains uniform beyond

$$\phi(-\infty) = \phi(\xi_c) = \beta. \tag{51}$$

The hose instability grows from random noise fluctuations in the region at ξ_c where ϕ just begins to exceed β, just as the instability grew from where ϕ just began to exceed zero in the previous section. Thus $\xi_c = \xi_1$ and

$$\phi_1 = \beta. \tag{52}$$

Now from (28) it is readily shown that

$$Q_q(\phi) \cong \frac{2\beta^{q-1}}{3(\beta - \alpha)}(\phi - \beta)^{3/2} + 0^{5/2}(\phi - \beta)$$

so that $Q_q(\phi)_1 = 0$, and $A_q = 0$ according to (29). Then (31) reduces to

$$\xi = -\int_{\phi}^{1} \frac{d\phi\, \phi^{q-1}}{(\phi - \alpha)Q(\phi)} \tag{53}$$

and the limit as $\phi \to \beta$ we have

$$\phi(\xi) \sim \beta + 9/\xi^2 + 0(1/\xi^3), \tag{54}$$

so that the precursor is infinitely long, $\xi_c = -\infty$. The characteristic scale of the precursor is

$$1/\lambda = (1 - \alpha)Q_q(1)\gamma. \tag{55}$$

Now from (23) and (24) we see that α and β may be approximated as

$$\beta \cong 2\alpha = [N_1/N_2(0)][s^2 + 0(s^4)]$$

for s even slightly less than one. For a Mach number as large as 2 we have that β is of the order of 10^{-1}. Thus β, and consequently $\phi(-\infty)$, are small quantities except in weak shocks. By expanding in powers of α and β, (28) becomes

$$Q_q(\phi) = [2/(2q - 1)]\phi^{q-\frac{1}{2}} + 0(s^4\phi^{q-5/2})$$

so that

$$\lambda = \left(\frac{2q - 1}{4}\right)\frac{U}{\omega}\left[\frac{N_1}{N_2(0)}\right]^{\frac{1}{2}} \times \{1 - \tfrac{1}{2}s^2[1 - N_1/N_2(0)] + 0(s^4)\}. \tag{56}$$

Thus, the scale of the shock depends upon Mach number only to terms $0(s^2)$. For all but the weakest shocks the scale given in (50) for infinite Mach number is applicable.

4. SUMMARY

We have shown that a long precursor is a direct consequence of the quasi-linear shock model. The precursor is simply the region of linear growth of the hose instability, so that the characteristic scale of the shock front is of the order of the ion and electron Larmor radii, U/ω. The precursor density far ahead of the steep portion of the front relaxes (as the inverse square of the distance) to the fraction, $(C_1/U)^2 = 1/(\text{Mach Number})^2$, of the supersonic stream density N_1. Thus, while the precursor density does not actually vanish at infinity, it is none the less very small for all but weak shocks. The existence of the precursor probably forms the simplest experimental test of the model.

As we stated in the beginning, the quasi-linear model is based on the postulate that (1) at least a *statistically stationary* shock structure exists, and that (2) there is a region of linear growth (ahead of the steep portion of the front) which is in some degree separate from the boiling disorder in the region of breaking non-linear waves. Assumption (2) is based on the simple argument that any evaporation of ions in the forward direction from the region of disordering will automatically lead to a precursor of some sort, which must then, in the statistically stationary case, conform pretty much to the form discussed above. The relative importance of the precursor in the total shock transition is given by the smallness of F in equation (18), and presumably can be determined eventually by experiment or by sufficiently detailed machine calculations.

VOL. 2 196

REFERENCES

BUNEMAN O. (1959) *Journal of Nuclear Energy* Part C: *Plasma Physics*. This issue, p. 119.

COLGATE S. A. (1957) University of California Radiation Laboratory Report UCRL 4829.

DAWSON J. (1959) *Proceedings of the International Plasma Physics Institute*, Seattle. To be published.

GARDNER C. S., GOERTZEL H., GRAD H., MORAWETZ C. S., ROSE S. and RUBIN H. (1959) AEC Research and Development Report, New York University, NYU-2538.

LONGMIRE C. (1957) Los Alamos Lecture Notes, *Plasma Physics*.

PARKER E. N. (1958) *Phys. Rev.* **112,** 1429.

PARKER E. N. (1959) *Astrophys. J.* **129**, 217 861.

PETSCHEK H. E. (1958) *Rev. Mod. Phys.* **30,** 966.

OL.
2
961

J. Nucl Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 154 to 157. Pergamon Press Ltd. Printed in Northern Ireland

ON THE RADIATION FROM CO-OPERATIVE PHENOMENA IN PLASMAS

C. OBERMAN
Project Matterhorn, Princeton University, Princeton, New Jersey

Abstract—Of prime importance in plasma investigations is the knowledge of the interaction of a bounded plasma with external magnetic fields. The present paper concerns itself with the general properties of the radiation due to plasma oscillations of bounded plasmas of single geometries situated in a strong magnetic field. The special problems of transmission, reflection and scattering due to plasma slabs and columns are also discussed as well as the problems of the response of the plasma to near fields in the vicinity of its surface. The effects of thermal spread in the particle velocity distribution are included in the presentation.

A CONSIDERABLE part of the diagnostics of plasma behaviour rests on measurements made outside the plasma. Further, most means of manipulating plasma invoke externally applied electromagnetic fields. However, almost all treatments of, say, plasma oscillations have been given for plasmas of infinite extent. Such treatments give information on the propagation of waves inside a plasma at distances far removed from external boundaries, but give no indication of the coupling between these waves and the electromagnetic fields outside the plasma. This coupling determines the radiation from, say, plasma oscillations, as well as the response of the plasma to externally applied fields, where the fields may be either wave fields or near fields produced by charges and currents near the plasma surface.

We propose to consider the oscillations of a bounded plasma situated in a strong magnetic field. We restrict ourselves at the outset to the discussion of cylindrical and slab geometries with the large magnetic field parallel to the cylinder generators. Further, we shall initially restrict ourselves to a highly idealized and elementary representation of the plasma and its environment, and then later show how some of the restrictive features of this model may be suppressed.

The limitations of this model are expressed in the following assumptions:

(a) The magnetic field is of such strength that motions perpendicular to it may be neglected.
(b) Thermal motions of the electrons are negligible. The plasma is cold.
(c) The electrons behave like a charged, continuous fluid.
(d) The ions constitute a uniform, fixed neutralizing background.
(e) The amplitude of oscillation is so small that the linearized equations of motion are applicable.
(f) The mass motion of the electrons in the unperturbed plasma vanishes.
(g) The number density is uniform in the unperturbed state.

If the constraining magnetic field is taken in the z-direction, the linearized equations of motion for the perturbation quantities in the plasma are as follows: Gaussian units are employed.

$$\mathbf{e}_z m \frac{\partial V_z}{\partial t} = -\mathbf{e}_z e E_z,$$

$$\frac{\partial N}{\partial t} + n \frac{\partial V_z}{\partial z} = 0,$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \mathbf{e}_z \frac{4\pi e n}{c} V_z$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{E} = -4\pi e N,$$

$$\nabla \cdot \mathbf{B} = 0.$$

Here m and $-e$ are the mass and charge of an electron respectively, V_z is the perturbation velocity of the electrons, N is the perturbation electron number density in excess of the equilibrium value n. Outside the plasma, equations (3) through (6) apply with n taken equal to zero.

The large zero-order magnetic field constrains electrons to move parallel to the plasma surface, and hence, no surface charge can accumulate. Therefore, all components of **E** are continuous at the plasma surface. Also, since the plasma density is finite, there can be no sheet currents at the plasma surface. Thus, all components of **B** are continuous. The only discontinuity is in the number density. Thus, by taking the zero-order magnetic field infinite, we have removed one of the nastier difficulties of the problem, a verisimilar representation of the surface phenomena, charge separation, etc.

In order to completely specify the problem boundary

conditions at large distances must be given for the fields such as periodicity or reflecting ones, which conserve energy. However later, in the treatment of the problem of radiation due to plasma oscillations, it is convenient to abandon these energy conserving boundary conditions at large distances and admit the presence of perfectly absorbing boundaries.

Most of the problems we shall consider are most easily treated once the structure of the normal modes is given. I shall discuss these briefly.

First of all, I would like to exhibit an orthogonality relation exhibited by these modes. It is not necessary to know their complete structure but only that they are of the form $A(x, y, z, t) = A(x, y) \exp i(\omega t + k_z z)$, where A represents any one of the field quantities **E**, **B**, V_z, N. These present considerations thus apply to both the cylinder and slab situations.

It is easy to show that all field quantities are determined when E_z is determined. If $E_{z1}(x, y)$ and $E_{z2}(x, y)$ are two normal modes of the system (with same k_z) and ω_1 and ω_2 are the corresponding normal frequencies, then we find the following orthogonality relation:

$$\frac{1}{c^2}(\omega_1^2 - \omega_2^2)\left[\int_{U^0+U^i} E_{z1}E_{z2}\,dx\,dy - \frac{\nu^2\Omega^2}{\omega_1^2}\frac{c^2}{\omega_2^2}\int_{U^i} E_{z1}E_{z2}\,dx\,dy\right] = 0.$$

$$\frac{1}{c^2}(\omega_1^2 - \omega_2^2)\left[\int_{U^0+U^i} E_{z1}E_{z2}\,dx\,dy + \frac{c^2\Omega^2 m^2}{n^2e^2}\int_{U^i} N_1N_2\,dx\,dy\right] = 0.$$

Here U^0 and U^i are the regions in a constant z-plane exterior and interior to the plasma, respectively, and Ω^2 is the square of the plasma frequency. This equation is also given in an alternative form which exhibits the particle contribution to the modes. These modes can be normalized so that the term in brackets is δ_{12}, the Kronecker delta, if at least one of the modes is discrete, or $\delta(1-2)$ if they are both members of the continuum.

If we continue the Fourier decomposition of these modes, say, for the slab situation, we soon arrive at the following dispersion relation: which relates the wave numbers inside and outside the plasma to the frequency,

$$(k_x^0)^2 + k_y^2 + k_z^2 = \omega^2/c^2$$

$$(k_x^i)^2 = (k_x^0)^2(1 - \Omega^2/\omega^2) - k_y^2\Omega^2/\omega^2$$

$$\Omega^2 = 4\pi ne^2/m.$$

Since ω is real, and since k_y and k_z must be for boundedness of the modes at infinity, it results that k_x must be real or pure imaginary. Thus four possible types of modes exist depending upon the signatures of the squares of the wave numbers $(k_x^i)^2$ and $(k_x^0)^2$. The general character of these modes is exhibited in Fig. 1. Because of the symmetry of the situation, the modes fall into either even or odd forms in x.

Case A is the situation obtained when $(k_x^i)^2$ and $(k_x^0)^2$ are both greater than zero and represents waves propagating in the x-direction both inside and outside the plasma. This situation is arrived at when $\omega^2 > \Omega^2$ and when the phase velocity along the surface exceeds the velocity of light c. (For ease of presentation we shall take $k_y = 0$ unless otherwise indicated.)

Case B, where $(k_x^i)^2 > 0 > (k_x^0)^2$, represents modes which propagate in the x-direction inside the plasma but not outside. These are waves trapped in the plasma. This situation obtains when $\omega^2 < \Omega^2$ and the phase velocity along the surface is less than c. In this case a period equation must be satisfied in virtue of the boundary conditions, which in turn restrict k_x^0, k_{xi} and ω to discrete values for a given k_y, k_z. This period equation is given below as well as the values of k_x^0, k_x^i, and ω for given k_z and frequencies near the plasma frequency.

For Case B (slab width $= 2\delta$),

$$\begin{Bmatrix}\tan\\ -\cot\end{Bmatrix}(k_x^i\delta) = |k_x^0|/k_x^i \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$$

If $k_y = 0$, $\omega^2 \simeq \Omega^2$ then with $\epsilon = \Omega - \omega$ we find

$$k_x^{0^2} = (\Omega^2/c^2 - k_z^2) + 2\Omega\epsilon/c^2$$

$$k_x^{i^2} = 2(\epsilon/\Omega)(k_z^2 - \Omega^2/c^2).$$

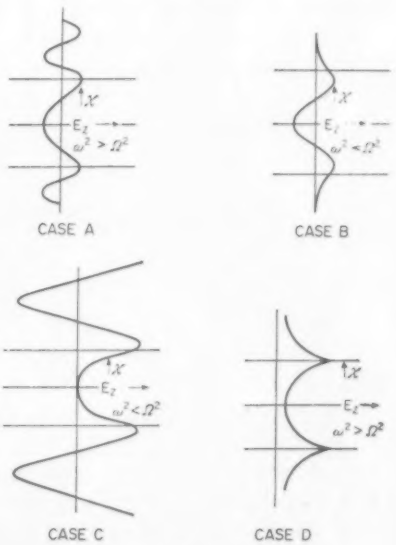


FIG. 1.

Then $\epsilon/\Omega \simeq$

$$\epsilon/\Omega \simeq \begin{Bmatrix}(n+1/2)^2\\ n^2\end{Bmatrix} \pi^2/2(k\delta+1)^2 \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}.$$

Case C represents modes which propagate outside but not inside. These are waves excluded from the plasma.

Modes of type D, which decay away from the plasma surface in both directions, are not obtained in the present problem because of the break in the derivative. Conditions other than here realized (surface charges) can give rise to modes of this type.

I would like now to touch briefly on several problems which can be treated using the previous normal mode analysis.

The first one I shall just mention is the scattering of an electromagnetic wave by a plasma cylinder. The previous normal mode analysis for the slab situation was repeated for the cylinder geometry with essentially similar results with respect to mode types and dispersion relation. These modes can be superposed so as to add up in the usual way to an incoming plane wave of unit amplitude plus an outgoing cylindrical wave. The amplitudes and phase shifts of the partial waves are derived as well as the differential and total cross-section. These results are functions of the plasma frequency, and the plasma radius, and may be of value in plasma diagnostics.

A second similar problem is that of the reflection and transmission of radiation for a plasma slab. That is, we consider the problem of plane wave with a given wave vector incident upon the slab and investigate the amplitudes of the transmitted and reflected waves. Again, superposition of the normal modes of this given k^0 so as to yield only an outgoing wave on the side opposite to incidence, gives the reflected and transmitted amplitudes and hence intensities. The transmission coefficients for Cases A and C (with k_y taken zero) are as follows:
Case A,

$$T = \left[1 + \frac{\Omega^4}{4\omega^4}(1-\Omega^2/\omega^2)^{-1}\sin^2(2k_x{}'\delta)\right]^{-1}.$$

Case C,

$$T = \left[1 + \frac{\Omega^4}{4\omega^4}(\Omega^2/\omega^2-1)^{-1}\sin h^2(2|k_x{}'|\delta)\right]^{-1}.$$

The usual interference properties of thin films are here realized. For instance, there is total transmission whenever $k_x{}'\delta = n\pi/2$.

Another interesting problem which we have considered was the formulation of the response of the plasma to an arbitrary distribution of sources on planes on either side of the plasma slab.

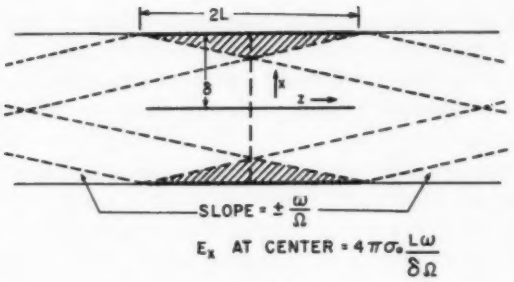


FIG. 2.

We have carried out in detail the problem of oppositely charge condenser plates of finite length placed close to the plasma surface and driven at a frequency $\omega \ll \Omega$. Fig. 2 shows schematically the geometry and standing wave pattern of the steady state behaviour of the field, which has strong maxima along the dotted lines. The shaded area shows the region of penetration of the vacuum field $4\pi\sigma_0 \exp i\omega t$. This possibility of getting strong low frequency fields into the plasma is an interesting one. Fig. 3 shows a plot of E_x at $z = 0$ as a function of x for various values of the parameter $l = L\omega/\delta\Omega$.

The last problem I wish to discuss is the radiation emitted by plasma oscillations of the slab. There are several approaches to the problem. We have formulated the initial-value of the problem in terms of normal modes (or equivalently by Laplace Transform) so that we can 'pluck' the plasma and watch the development of the radiation field in time. We have also formulated the problem of the response of the plasma to the presence of externally driven currents and charges and mechanical forces. Again, under these circumstances we can sit back and watch the

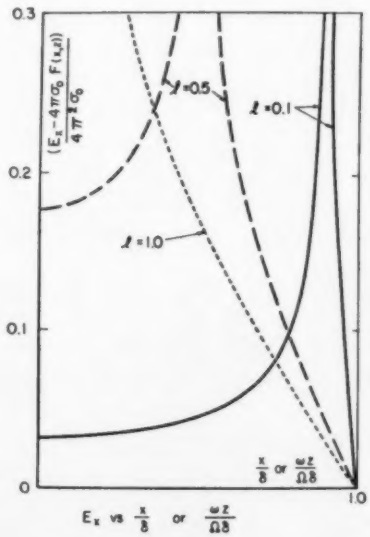


FIG. 3.

evolution and long-time behaviour of the field. A third method which yields results in close accord with the other two is to merely alter the boundary conditions at large distances so as to absorb all incident radiation.

The analysis now is formally similar to the other normal mode cases except now ω is to be taken complex with positive imaginary part to conform with the absorbing boundary condition. This in turn demands k_x^0 and k_x^i to be complex and the real parts of k_x^0 and ω must have the same sign (for $x > 0$) so as to yield only outgoing radiation. The period equation demanded by boundary conditions at the plasma surface is:

$$\begin{Bmatrix} \tan \\ -\cot \end{Bmatrix} (k_x^i \delta) = i k_x^0 k_x^i \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}.$$

Let $\omega = \Omega + \epsilon$

$$\text{then} \quad \epsilon/\Omega = \begin{Bmatrix} (n + 1/2)^2 \\ n^2 \end{Bmatrix} \frac{\pi^2}{2(k^2\delta^2 + 1)^2} \times (k^2\delta^2 - 1 + 2ik\delta) \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$$

where $k = (\Omega^2/c^2 - \nu^2)^{1/2}$.
Also

$$k_x^0 = k + \frac{\Omega^2}{kc^2} \frac{\epsilon}{\Omega}.$$

For the case $k_y = 0$ and ω close to Ω we exhibit the characteristic frequencies and wave numbers and the explicit exponentiation in space and time. Fig. 4 shows the form of these modes.

At this time I would like to speak about our attempts to remove the restrictive assumptions inherent in our plasma model.

The first attempt at generalization was to shrink the constraining magnetic field to a finite value. The plasma was kept cold, the equilibrium density uniform, but the ion dynamics were now also included. The particles can now execute significant motions perpendicular to the constraining field and one must face up to a description of the particle behaviour at the plasma boundary. As a simple and tractable model we chose the enveloping region to be one which was optically transparent and materially impervious but permitting the accumulation of surface charges and currents. These charges and currents are taken related to the internal motion of the plasma by the equation of continuity. This model should then describe to some degree, the effect of surface charge accumulation on the penetration of electromagnetic radiation into the plasma and escape of radiation from the plasma.

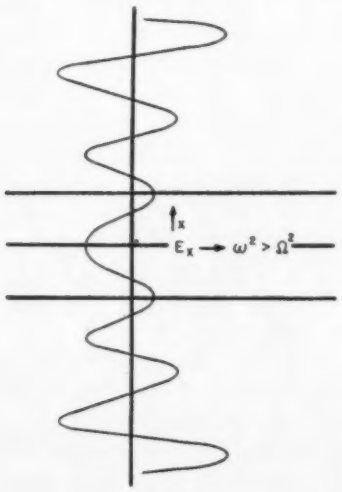


FIG. 4.

Again the normal modes were derived as in the previous analysis together with the concomitant dispersion relation which involves now both ion and electron gyrofrequencies as well as both plasma frequencies as parameters. The overall character of these modes is much the same as that given previously except there are now situations in which the mode structure internal to the plasma is of a growing or damped oscillatory variety rather than pure decaying or oscillatory.

We have repeated the treatment of special problems given for the more elementary model the only difficulty being one of algebraic complexity. I shall not quote any results at this time since some of these results are not yet complete and those which are complete have not been totally gleaned of their information. I only wish to remark that in addition to the intensity effects described on the earlier simple model there is evidence of many interesting polarization phenomena which might be of diagnostic value.

Surindar Trehan is currently looking at the model where the magnetic field is restored to its infinite value but where the equilibrium number density is now no longer uniform and the ion dynamics are taken into account. In particular for the slab situation he finds under the assumption of parabolic profile that the problem is formally equivalent to the one-dimensional quantum mechanical oscillator and thus soluble in terms of Hermite Polynomials (or more generally in terms of hypergeometric functions). He has not yet completed his consideration of special problems involving these modes.

In closing I would like to state that we have seized the bull by the horns and formulated the problem with the full Vlasov set of equations as description (with the equilibrium number density uniform). As yet no problems as those previously considered have been worked out.

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 158 to 163. Pergamon Press Ltd. Printed in Northern Ireland

NON-LINEAR EFFECTS IN ELECTRON PLASMAS*

P. A. STURROCK
Microwave Laboratory, W. W. Hansen Laboratories of Physics, Stanford University, Stanford, California

Abstract—The effect of non-linear terms in the dynamical equations governing wave propagation in plasmas may be analysed by a perturbation procedure which is acceptable for amplitudes which are not too large. The Hamiltonian describing the complete system is separated into two parts: the quadratic part which yields the linearized equations and the non-linear part. The quadratic part may be eliminated by a normal-mode analysis, the 'normal modes' comprising travelling waves. The non-linear part then results in interaction between these waves.

Two theorems concerning wave interaction are proved. The first relates energy-transfer between a group of interacting waves to the frequency of these waves. These 'action-transfer relations' lead to the MANLEY-ROWE relations for steady-state or quasi-steady-state configurations. The second theorem relates the frequency-displacements of a group of interacting waves to the energies of these waves.

The properties of electron plasmas undergoing longitudinal oscillations are re-examined in the light of the preceding theorems. Interaction terms may be classed as 'coherent' and 'incoherent': the former do not result in energy transfer but only frequency displacement which may be characterized by a dispersion relation. The second group leads to transfer of energy between waves and hence to spectral decay.

The interaction between longitudinal (electrostatic) and transverse (electromagnetic) waves in plasmas is considered and it is shown that in a uniform plasma in the absence of magnetic fields, the dominant interaction couples two longitudinal waves with one transverse wave. Hence one would anticipate that the dominant non-linear mechanism for radiation from excited plasmas leads to emission at twice the plasma frequency.

VOL
2
196

1. INTRODUCTION

IN his talk entitled 'Instabilities and Growing Waves in Electron Beam Devices,' PIERCE (1959) explained two important and fruitful concepts in the theory of electron tubes, that of the coupling of modes (PIERCE, 1954) and the MANLEY-ROWE relations (MANLEY and ROWE, 1956). These concepts have general importance in the theory of wave-propagating systems and we shall see how they apply in particular to the problem of non-linear effects in electron plasmas.

It is convenient, in approaching this problem, to begin with a very general view. We consider the problem of analysing the non-linear effects in an arbitrary wave-propagating or oscillatory system which we choose to describe by a Hamiltonian. Separation of the Hamiltonian enables us to interpret this problem as that of the interaction of waves. These interaction terms may be analysed into a sequence involving two, three, four, . . . waves at a time, in such a way that terms involving small numbers of waves are dominant at small amplitude levels.

By considering one interaction term at a time, we may establish general relations concerning the exchange of energy between interacting modes, and the shift of frequency of a wave due to interaction of the waves. The former set of relations, which are termed 'action-transfer relations,' may, in a suitably restricted problem, be combined to yield the MANLEY-ROWE relations (STURROCK, 1960).

In applying this theory to discussion of non-linear effects in plasmas, we consider for simplicity a plasma of zero temperature and construct the appropriate Hamiltonian by means of the Lagrangian which describes this system (STURROCK, 1958). As is well-known, normal modes of the electron plasma separate into acoustic or longitudinal modes of the type originally discussed by LANGMUIR (TONKS and LANGMUIR, 1929) and optical or transverse modes which represent propagating electromagnetic waves (SPITZER, 1957). The effect of interaction among acoustic modes, which has been considered in detail elsewhere (STURROCK, 1957) is reviewed here briefly. In conclusion, we consider the problem of the radiation of electromagnetic energy by a plasma undergoing plasma oscillations by considering the coupling of acoustic to optical modes.

2. WAVE-INTERACTION FORMALISM

The study of wave-propagating systems may be conceptually simplified by the familiar artifice of 'box normalization' which reduces the number of normal modes from a non-denumerably infinite set to a

* This research was supported in whole or in part by the United States Air Force under Contract No. AF 49(638)-415 monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

denumerably infinite set. In the linear approximation, this enables one to interpret the system as an assembly of harmonic oscillators which may be described by a Hamiltonian:

$$H^0 = \sum_\lambda \tfrac{1}{2}(p_\lambda^2 + \omega_\lambda^2 q_\lambda^2), \tag{2.1}$$

The complete Hamiltonian, however, must be written as

$$H = H^0 + H^{\mathrm{I}}, \tag{2.2}$$

wherein H^{I} comprises non-linear terms. (One may also choose to include in H^{I} small coupling terms if the normal modes are thereby simplified.)

The term H^{I} may be conveniently expressed as a sequence of polynomials, of third, fourth, etc., order in the variables p_λ, q_λ. Each term of each polynomial therefore represents the interaction of a certain number of waves of the linear system, so that the problem has been conceptually simplified; it may also have been simplified computationally if we find that only certain of these interaction terms need be considered in a particular problem.

It is convenient to describe the system by action-angle variables (GOLDSTEIN, 1956), θ_λ, J_λ, so that H^0 becomes

$$H^0 = \sum_\lambda \omega_\lambda J_\lambda. \tag{2.3}$$

We obtain the usual relation between action, frequency and energy of the oscillator:

$$E_\lambda = J_\lambda \omega_\lambda. \tag{2.4}$$

A further transformation to variables J_λ, κ_λ,

$$\theta_\lambda = \omega_\lambda t + \kappa_\lambda, \tag{2.5}$$

enables us to eliminate H^0 completely. The interaction Hamiltonian may now be expressed as

$$H^{\mathrm{I}} = \sum_{\substack{L l \lambda \\ M m \mu}} F_{L l \lambda M m \mu \ldots} J_\lambda^{\frac{1}{2}L} J_\mu^{\frac{1}{2}M} \ldots \exp il(\omega_\lambda t + \kappa_\lambda) \times \exp im(\omega_\mu t + \kappa_\mu) \tag{2.6}$$

where $L, M, \ldots$, take positive integral values and $l, m, \ldots$, integral values of either sign restricted by

$$|l| \leqslant L, \quad L + l = \text{even integer, etc.} \tag{2.7}$$

3. GENERAL RELATIONS GOVERNING WAVE INTERACTION

In terms of the variables J_λ, κ_λ, the canonical equations for the system take the form

$$\frac{dJ_\lambda}{dt} = -\frac{\partial H^{\mathrm{I}}}{d\kappa_\lambda}, \quad \frac{d\kappa_\lambda}{dt} = \frac{\partial H^{\mathrm{I}}}{\partial J_\lambda}. \tag{3.1}$$

Since the right-hand sides of these two equations may be expressed as the sum of contributions by means of the expansion (2.6), we may consider the contribution arising from just one interaction term. If we consider, in particular, the term $Ll\lambda Mm\mu \ldots$, the first equation of (3.1) leads to the following relations,

$$\frac{(dJ_\lambda/dt)_e}{l} = \frac{(dJ_\mu/dt)_e}{m} = \ldots \overset{\text{amp}}{=} E_e, \tag{3.2}$$

where E_e is the energy of the interaction term, and the last equality is intended to denote an equality in amplitude only as time or the relevant phases are varied. Equation (2.4) enables us to rewrite (3.2) as

$$\frac{(dE_\lambda/dt)_e}{l\omega_\lambda} = \frac{(dE_\mu/dt)_e}{m\omega_\mu} = \ldots \overset{\text{amp}}{=} E_e. \tag{3.3}$$

If we regard the interaction term under consideration as representing the wave λ, taken $|l|$ times over, and the wave μ, taken $|m|$ times over, etc., then we see that equation (3.2) implies that the rate of transfer of action to or from each mode taking part in the interaction is the same for all modes: the direction of transfer depends on the sign of the coefficients l, m, etc. This relation is strongly reminiscent of the quantum-mechanical theory of particle collisions, since the change of particle numbers in a state is measured by the change in action of that state. This analogy is strengthened if we note that cumulative interaction is to be attributed to interaction terms which obey the selection rule

$$l\omega_\lambda + m\omega_\mu + \ldots = 0. \tag{3.4}$$

Equation (2.4), when coupled with the relation (3.2), shows that this may be interpreted as an equation of conservation of energy. In the theory of wave-propagating systems, the further selection rule

$$l\mathbf{k}_\lambda + m\mathbf{k}_\mu + \ldots = 0 \tag{3.5}$$

will arise which leads to an analogous equation of conservation of momentum since momentum $\mathbf{P}_\lambda$ of each wave is related to the wave number $\mathbf{k}_\lambda$ by

$$\mathbf{P}_\lambda = J_\lambda \mathbf{k}_\lambda. \tag{3.6}$$

If the behaviour of a system is periodic, energy possibly being supplied and removed by external means, or if the behaviour is quasi-periodic, the energy exchange being slow by comparison with the appropriate frequencies, we may form certain relationships involving the total interaction process rather than individual contributions to the interaction. Suppose that all frequencies excited in the system may be composed from two or more incommensurable frequencies $\omega_a, \omega_b, \ldots$, as follows

$$\omega_{rs\ldots} = r\omega_a + s\omega_b + \ldots, \tag{3.7}$$

where $r, s, \ldots$ take positive or negative integral values. Equation (3.3) may then be written as

$$\frac{(dE_{rs}/dt)_c}{r\omega_a + s\omega_a + \ldots} = \frac{(dE_{r's'\ldots}/dt)_c}{r'\omega_a + s'\omega_b + \ldots} = \ldots = C_{rs\ldots r's'\ldots} \qquad (3.8)$$

for each interaction term where for convenience we suppress all suffixes except $r, s, \ldots$, etc. Clearly $C_{rs\ldots r's'\ldots\ldots}$ is unchanged under permutation of sets of suffixes $rs\ldots$; $r's'\ldots$; etc. Using this fact, and noting that (3.4) requires that

$$\left.\begin{aligned} r + r' + \ldots &= 0, \\ s + s' + \ldots &= 0, \\ \ldots &\quad , \end{aligned}\right\} \qquad (3.9)$$

and summing over all coupling terms, we find that

$$\sum_{rs\ldots} r \frac{dE_{rs\ldots}/dt}{r\omega_a + s\omega_b + \ldots} = 0, \text{ etc.} \qquad (3.10)$$

These are generalized forms of the MANLEY-ROWE relations (MANLEY and ROWE, 1956) which were originally derived for electrical networks.

If we now consider the second of the canonical equations (3.1), and consider one of the interaction terms only, we arrive, by analogy with the relations (3.2), at the relations

$$\frac{E_\lambda}{L}\frac{\Delta_c\omega_\lambda}{\omega_\lambda} = \frac{E_\mu}{M}\frac{\Delta_c\omega_\mu}{\omega_\mu} = \ldots = \frac{1}{2}E_c \qquad (3.11)$$

where we have chosen to interpret the contribution to $d\kappa_\lambda/dt$ as a shift in the frequency ω_λ. These relations, which may be termed 'frequency shift relations,' relate the shifts of frequencies of a group of interacting modes to the partition of energy among these modes.

4. INTERACTION OF ACOUSTIC (LONGITUDINAL) MODES

As is well-known, waves in electron plasmas (in which the motion of ions is neglected) which are infinite, cold and free from 'external fields' may be analysed into two types: the acoustic or longitudinal waves studied by LANGMUIR and the optical or transverse electromagnetic waves. Non-linear interaction among the former waves alone has been studied elsewhere (STURROCK, 1957) by methods different from those here proposed. However, we may briefly review the results and note that they are compatible with the general relations established in the previous section.

Since the dispersion relation characteristic of acoustic modes is

$$\omega^2 = \omega_P^2, \qquad (4.1)$$

where ω_P is the plasma frequency, we see that interacting groups of waves, which must satisfy the relations (3.4) and (3.5), must involve an even number of waves. Hence the lowest order non-linear effect is of fourth order such that the selection rules reduce to

$$\omega_P + \omega_P = \omega_P + \omega_P, \qquad (4.2)$$

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4. \qquad (4.3)$$

We see at once from (3.3) that the action-transfer relations lead to the following contributions to the rates of change of energy of these interacting waves:

$$\left(\frac{dE_1}{dt}\right)_c = \left(\frac{dE_2}{dt}\right)_c = -\left(\frac{dE_3}{dt}\right)_c = -\left(\frac{dE_4}{dt}\right)_c. \qquad (4.4)$$

This relation was already noted in STURROCK (1957), although the origin of this relation was not clear.

Among the groups of waves satisfying (4.3), we should give particular attention to those for which $\mathbf{k}_1 = \mathbf{k}_3$ (or $\mathbf{k}_4$) so that $\mathbf{k}_2 = \mathbf{k}_4$ (or $\mathbf{k}_3$). It follows at once from (4.4) that this interaction does not give rise to an exchange of energy between the waves $\mathbf{k}_1$ and $\mathbf{k}_2$: this has been termed 'coherent interaction' and it has been found that interaction of this type may be characterized by a dispersion relation. That there is an effective shift of frequency due to interaction between modes is, of course, a consequence of the frequency-shift relations: indeed, one could construct the appropriate dispersion relation by means of (3.11).

Let us consider briefly one-dimensional spectra, for which all wave numbers are in the same direction. We may now make use of the analogy between non-linear interaction between groups of waves and collisions between particles, which was noted in the previous section. If two like particles collide elastically, moving in the same direction, they merely exchange kinetic energy and momentum. This indicates that the fourth-order interaction between waves of a one-dimensional spectrum will be of the type termed 'coherent.' This implies that there will be no 'spectral decay,' by which we mean transfer of energy between modes which leads ultimately to the dissipation of ordered motion into disordered motion. This conclusion is in agreement with the results of studies of one-dimensional excitations of electron plasmas (DAWSON, 1958).

The above considerations imply that the lowest order interaction which gives rise to spectral decay is of the fourth order provided that the spectrum is not one-dimensional. Hence if a wave spectrum is almost one-dimensional, the dominant non-linear effect will be that which renders the spectrum isotropic. It has been found, by consideration of a particular

model (STURROCK, 1957), that the number of plasma oscillation periods in which an almost one-dimensional spectrum tends towards isotropy is expressible as

$$N = \frac{1}{4\epsilon D^2}, \tag{4.5}$$

where ϵ is the half-angle of the spectrum and D is the depth of modulation defined as $\langle \Delta\rho \rangle / \rho$, the fractional change of electron density.

5. COUPLING BETWEEN ACOUSTIC AND OPTICAL MODES

We may set up the Hamiltonian describing vibrations of a cold electron plasma by means of the action principle. Calculations are simplified if we use 'natural' or 'electron-optical' (STURROCK, 1955) units.* If the unperturbed plasma density is ρ, if the electromagnetic field is described by the magnetic vector potential A_r (the gauge so chosen that $\phi = 0$), and if the electron motion is characterized by the displacement vector $\xi_r(x, t)$, the Lagrangian density is given by

$$\mathscr{L} = \frac{1}{2}\rho\dot{\xi}_r^2 + \frac{1}{2}\left(\dot{A}_r^2 - \frac{\partial A_r}{\partial x_s}\frac{\partial A_r}{\partial x_s} + \frac{\partial A_r}{\partial x_s}\frac{\partial A_s}{\partial x_r}\right) - \rho\dot{\xi}_r A_r(\mathbf{x} + \boldsymbol{\xi}). \tag{5.1}$$

If we now 'box normalize' in the usual way, we may carry out Fourier transformation as follows:

$$\xi_r(\mathbf{x}, t) = \sum_{\mathbf{k}} \xi(\mathbf{k}, t) \exp i\mathbf{k}\cdot\mathbf{x}, \text{ etc.} \tag{5.2}$$

Then we find that the second and third-order polynomials in the expansion of the Lagrangian function are

$$L^{(2)} = V[\tfrac{1}{2}\rho\dot{\xi}_r\dot{\xi}_r^+ + \tfrac{1}{2}(\dot{A}_r\dot{A}_r^+ - k^2A_rA_r^+ + k_rk_sA_rA_s^+) - \rho\dot{\xi}_rA_r^+], \tag{5.3}$$

$$L^{(3)} = -V\rho\{\dot{\xi}_r, \xi_s, ik_sA_r\}_0, \tag{5.4}$$

where we introduce the notation

$$\xi_r^+(\mathbf{k}, t) = \xi_r(-\mathbf{k}, t), \tag{5.5}$$

and

$$\{f, g\} = \sum_{\mathbf{k}'} f(\mathbf{k} - \mathbf{k}')g(\mathbf{k}'),$$

$$\{f, g, h\} = \sum_{\mathbf{k}', \mathbf{k}''} f(\mathbf{k} - \mathbf{k}' - \mathbf{k}'')g(\mathbf{k}')h(\mathbf{k}''), \text{ etc.}, \tag{5.6}$$

and denote by the suffix 0 a convolution evaluated for $\mathbf{k} = 0$.

We may now form the conjugate variables:

$$\tilde{\omega}_r \equiv \frac{\partial L}{\partial\dot{\xi}_r} = V\rho[(\dot{\xi}_r^+ - A_r^+) - \{\xi_s, ik_sA_r\}^+ + \ldots], \tag{5.7}$$

$$P_r \equiv \frac{\partial\mathscr{L}}{\partial\dot{A}_r} = V\dot{A}_r^+. \tag{5.8}$$

These formulae enable us to construct the Hamiltonian, of which we find the second and third-order polynomial contributions to be

$$H^{(2)} = \tfrac{1}{2}V^{-1}\rho^{-1}\{\tilde{\omega}_r, \tilde{\omega}_r\}_0 + \tfrac{1}{2}V^{-1}\rho^{-1}\{P_r, P_r\}_0 + \{\tilde{\omega}_r^+, A_r\}_0 + \tfrac{1}{2}V\rho\{A_r, A_r\}_0 + \tfrac{1}{2}V\{k^2A_r, A_r\}_0 + \tfrac{1}{2}V\{k_rA_r, k_sA_s\}_0, \tag{5.9}$$

$$H^{(3)} = \{\tilde{\omega}_r^+, \xi_s, ik_sA_r\}_0 + V\rho\{A_r, \xi_s, ik_sA_r\}_0. \tag{5.10}$$

One may solve the linear system by deriving the canonical equations from (5.9). These lead to the equations

$$\ddot{\xi}_r - \dot{A}_r = 0 \tag{5.11}$$

and

$$\ddot{A}_r + \rho A_r + k^2A_r - k_rk_sA_s = 0. \tag{5.12}$$

The solutions of (5.12) fall into two classes: the acoustic or longitudinal modes for which

$$\left.\begin{aligned} A_r(\mathbf{k}, t) &= \mathscr{A}_L(\mathbf{k})l_r \exp i\omega t, \\ \xi_r(\mathbf{k}, t) &= -i\omega^{-1}\mathscr{A}_L(\mathbf{k})l_r \exp i\omega t, \\ P_r(\mathbf{k}, t) &= -i\omega V\mathscr{A}_L^+(\mathbf{k})l_r \exp -i\omega t, \\ \tilde{\omega}_r(\mathbf{k}, t) &= 0, \end{aligned}\right\} \tag{5.14}$$

$$\omega^2 = \omega_p^2 \equiv \rho, \tag{5.13}$$

and the optical or transverse modes for which

$$\omega^2 = \omega_p^2 + k^2, \tag{5.15}$$

and

$$\left.\begin{aligned} A_r(\mathbf{k}, t) &= \mathscr{A}_T(\mathbf{k})e_r \exp i\omega t, \\ \xi_r(\mathbf{k}, t) &= -i\omega^{-1}\mathscr{A}_T(\mathbf{k})e_r \exp i\omega t, \\ P_r(\mathbf{k}, t) &= -i\omega V\mathscr{A}_T^+(\mathbf{k})e_r^+(\mathbf{k}) \exp -i\omega t, \\ \tilde{\omega}_r(\mathbf{k}, t) &= 0, \end{aligned}\right\} \tag{5.16}$$

where we have denoted by $\mathbf{l}$ the unit vector parallel to $\mathbf{k}$ and by $\mathbf{e}$ the polarization vector of the transverse wave, so that

$$|\mathbf{k}|\,\mathbf{l} = \mathbf{k}, \quad \mathbf{k}\cdot\mathbf{e} = 0, \quad |\mathbf{e}| = 1. \tag{5.17}$$

We find from

$$J = \frac{1}{2\pi}\oint (P_r\,dA_r + \tilde{\omega}_r\,d\xi_r) \tag{5.18}$$

that the action to be attributed to either wave is given by

$$J = V\omega\,|\mathscr{A}|^2. \tag{5.19}$$

* Unit of length 1 cm; time, $3{\cdot}33 \times 10^{-11}$ sec; electric potential 511,200 volts; magnetic field strength, 1,704 Gauss; current, 1,356 amp; charge density, $2{\cdot}834 \times 10^{11}$ electronic charges per cm^3.

Let us now consider the particular problem of the coupling of longitudinal and transverse modes. One aspect of the interest of this problem is to evaluate the radiation to be expected from a spectrum of plasma oscillations, that is of acoustic modes. The lowest order interaction will occur at third order, and will involve two longitudinal modes of frequency ω_P and wave numbers $\mathbf{k}_1$, $\mathbf{k}_2$, and the transverse mode of frequency $2\omega_P$ and wave number $\mathbf{k}_1 + \mathbf{k}_2$. It is interesting to note that this mechanism gives rise to radiation at twice the plasma frequency rather than at the plasma frequency. Denote by $\mathscr{A}_1$, $\mathscr{A}_2$, the amplitudes to be assigned to the longitudinal modes and by $\mathscr{A}_T$ the amplitude to be assigned to the transverse mode. We may evaluate the coupling between these modes by calculating the appropriate contribution to $H^{(3)}$. We find, from (5.10), (5.14) and (5.16) that this contribution is expressible as

$$H^{(3)} = V\omega_p \mathscr{A}_1 \mathscr{A}_2 \mathscr{A}_T^* (\mathbf{e} \cdot \mathbf{l}_1 |\mathbf{k}_2| + \mathbf{e} \cdot \mathbf{l}_2 |\mathbf{k}_1|). \quad (5.20)$$

On noting that $\mathbf{e}$ is normal to $\mathbf{k}_1 + \mathbf{k}_2$, we see that the coupling vanishes if $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel, and that it vanishes if $|\mathbf{k}_1| = |\mathbf{k}_2|$.

We find from (3.1), (5.19) and (5.20) that

$$\mathscr{A}_T = -\tfrac{1}{2} i \mathscr{A}_1 \mathscr{A}_2 (\mathbf{e} \cdot \mathbf{l}_1 |\mathbf{k}_2| + \mathbf{e} \cdot \mathbf{l}_2 |\mathbf{k}_1|), \quad (5.21)$$

an expression which, it should be noted, does not depend upon the method of box normalization. In order to obtain a numerical estimate of the effect which this equation represents, it is convenient to note that the phase velocities of the longitudinal waves are likely to be small with respect to the velocity of light, since the phase velocity will be the same as the velocity of the electron beam exciting these waves. Hence it is appropriate to write

$$\mathbf{k}_1 = \omega_P \frac{\mathbf{v}}{v^2}, \quad \mathbf{k}_2 = -\omega_P \frac{\mathbf{v} + \Delta\mathbf{v}}{|\mathbf{v} + \Delta\mathbf{v}|^2}, \quad (5.22)$$

where the velocities of the two beams exciting the plasma oscillations are taken to be $\mathbf{v}$ and $-(\mathbf{v} + \Delta\mathbf{v})$. On noting from (5.15) that

$$(\mathbf{k}_1 + \mathbf{k}_2)^2 = 3\omega_P^2, \quad (5.23)$$

we see that $\Delta\mathbf{v}$ must be small by comparison with $\mathbf{v}$; in fact,

$$|\Delta\mathbf{v}|^2 \approx 3v^4. \quad (5.24)$$

The problem now remains of maximizing the expression appearing in (5.21) subject to the restriction that

$$\mathbf{e} \cdot (\mathbf{k}_1 + \mathbf{k}_2) = 0. \quad (5.25)$$

These conditions require that $\mathbf{e}$ and $\Delta\mathbf{v}$ should be parallel, inclined at 45° to the vector $\mathbf{v}$. We then find that

$$|\mathscr{A}_T| = (\sqrt{3}/2)\omega_p |\mathscr{A}_1| |\mathscr{A}_2|. \quad (5.26)$$

On noting that

$$|\mathbf{A}| = \omega_p |\boldsymbol{\xi}|, \quad |\mathbf{k}| |\boldsymbol{\xi}| = \frac{\langle \Delta\rho \rangle}{\rho}, \quad (5.27)$$

we find that the number of plasma oscillations N in which an appreciable fraction of the energy is transfered from longitudinal to transverse modes is given by

$$N \approx \frac{1}{5} \frac{1}{(v/c)(\Delta\rho/\rho)}, \quad (5.28)$$

where we have returned to conventional units. If, for example, the depth of modulation $\Delta\rho/\rho$ is 1 per cent and $v/c = 0{\cdot}02$, corresponding to an electron beam energy of 100 V, we find that it takes 1,000 plasma oscillations for an appreciable transfer of energy to take place.

6. DISCUSSION

The wave-interaction formalism which we have developed for treating non-linear effects in wave-propagating systems is useful both for the conceptual simplification which it offers and for the concomitant procedures for computing various processes intrinsic to a non-linear medium. Thus we have seen that concepts which have been applied in the coupled-mode theory of electron tubes are relevant also to more complicated systems such as plasmas. Mechanisms such as those underlying parametric amplifiers can occur also in electron plasmas. In particular, we have seen that the action-transfer relations which have been so useful in the theory of non-linear electronic systems are applicable to a very wide class of physical systems including, in particular, plasmas.

The example given in Section 5 of the coupling of acoustic and optical modes in a plasma provides an example of the computational usefulness of the wave-interaction formalism. The result we have noted, that the dominant interaction mechanism produces radiation at twice the plasma frequency, may well explain a peculiarity of observations of the generation of solar radio noise. Observations (WILD, MURRAY and ROWE, 1954) indicate that radiation is emitted at the plasma frequency itself and at the second harmonic, but there is little evidence for appreciable generation at higher harmonics. Our calculations indicate why generation at the second harmonic should be stronger than that at any higher harmonic. Non-linear effects will give rise to radiation at the fundamental only at an order higher than that giving rise to radiation at the second harmonic. However, inhomogeneities of the

VOL 2 196

plasma or a magnetic field would promote coupling between acoustic and optical modes at the fundamental frequency, so giving rise to radiation. It seems, indeed, that radiation observed at the fundamental is usually associated with frequency-splitting such as might be attributed to a magnetic field, whereas radiation is stronger at the second harmonic than at the fundamental if there is no marked splitting. Although inhomogeneities also would give rise to emission at the plasma frequency (without frequency-splitting), one would not expect this to be as strong as emission due to the presence of a magnetic field. These mechanisms call for further calculation, but our results do go some way towards explaining why emission is observed predominantly at the fundamental and at the second harmonic.

REFERENCES

DAWSON J. M. (1958) Project Matterhorn Report No. PM–S–36, NYO–8050. Princeton University.

GOLDSTEIN H. (1956) *Classical Mechanics* p. 288 *et seq.* Addison-Wesley, Cambridge, Mass.

MANLEY J. M. and ROWE H. E. (1956) *Proc. I.R.E.* **44,** 904.

PIERCE J. R. (1954) *J. Appl. Phys.* **25,** 179.

PIERCE J. R. (1960) *Journal of Nuclear Energy* Part C: *Plasma Physics.* This issue pp. 73, 135.

SPITZER L. (1957) *Physics of Fully Ionized Gases,* p. 50 *et seq.* Interscience, New York.

STURROCK P. A. (1955) *Static and Dynamic Electron Optics,* pp. 3–6, Cambridge University Press.

STURROCK P. A. (1957) *Proc. Roy. Soc.* **A242,** 277.

STURROCK P. A. (1958) *Ann. Phys.* **4,** 306.

STURROCK P. A. (1960) *Ann. Phys.* **9,** 422–434.

TONKS L. and LANGMUIR I. (1929) *Phys. Rev.* **33,** 195.

WILD J. P., MURRAY J. D. and ROWE W. C. (1954) *Aust. J. Phys.* **7,** 439.

VOL.
2
1961

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 164 to 168. Pergamon Press Ltd. Printed in Northern Ireland

EXPERIMENTAL STUDY OF A PLASMA COLUMN IN A MICROWAVE CAVITY*

S. J. BUCHSBAUM and E. I. GORDON
Bell Telephone Laboratories, Inc., Murray Hill, New Jersey

and

S. C. BROWN
Massachusetts Institute of Technology, Cambridge, Massachusetts

Abstract—Experiments designed to study the production of a steady-state plasma column by microwave cavity means are described. At low plasma densities electrons are heated by cyclotron resonance in crossed microwave electric and static magnetic fields. Phenomena associated with large energies which the electrons possess near cyclotron resonance are discussed. Large plasma densities are achieved by resonating the plasma column by suitably varying the static magnetic field, the microwave frequency and the input power. In this manner, densities of the order of 10^{12} cm^{-3} are obtained at a neutral gas pressure in the micron range.

INTRODUCTION

THE experiments to be described in this paper were undertaken with the aim of producing, by microwave means, a steady-state, highly-ionized plasma of such density that it could be studied by microwave cavity techniques. These techniques are described in detail elsewhere (BUCHSBAUM, 1958), where it is shown that they are applicable to measurement of densities as high as 10^{12} to 10^{13} cm^{-3}. This implies that in order to obtain measurable high degrees of ionization, the plasma must be produced at neutral gas pressures in the micron range. At such low pressures the electron mean free path is large and Joule heating is ineffective. Consequently, resonance heating was resorted to; cyclotron resonance for gas breakdown and plasma resonance for the density build-up.

EXPERIMENTAL

Fig. 1 shows a schematic of the heart of the apparatus. The plasma is produced in a cylindrical quartz tube 1 cm in diameter and 6 cm long. The tube is placed coaxially in a cylindrical microwave cavity which, in turn, is held between the pole pieces of an electromagnet. The pole pieces are provided with shims in order to increase the uniformity of the magnetic field over the cross-section of the quartz tube. The tube could be filled to any desired pressure with gas (usually hydrogen) of high purity.

The cavity is driven from a 50-watt, variable-frequency S-band magnetron in a TE_{111} mode. Because the diameter of the quartz tube is small when compared with the diameter of the cavity, the electric field of TE_{111} mode is uniform over the cross-section of the tube to better than one per cent.

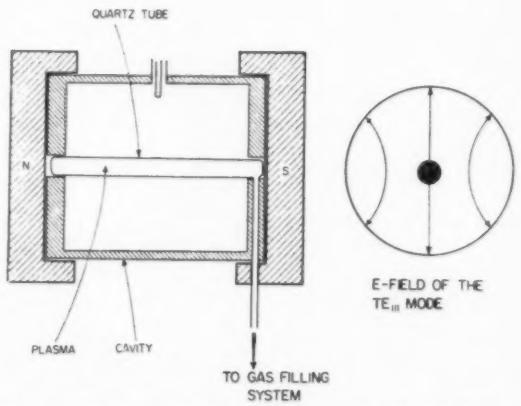


FIG. 1.—Schematic diagram of apparatus and the electric field lines of the TE_{111} mode.

(a) *Low electron densities*

At low plasma densities, where the microwave field is not perturbed by the plasma, a charged particle finds itself in a crossed (uniform) microwave electric field and a static magnetic field. When the microwave frequency, ω, is made equal to the electron cyclotron frequency, $\omega_b = eB/m$, a very efficient mechanism is provided for transfer of energy from the microwave field to the electron, provided the electron can complete many orbits before colliding. This condition is well satisfied in our experiment. Thus, the electron energy is largest at cyclotron resonance and, indeed, we find that breakdown curves exhibit a minimum at $\omega = \omega_b$. See Fig. 2.

* This work was performed at the Research Laboratory of Electronics of the Massachusetts Institute of Technology, Cambridge, Massachusetts, and was supported in part by the U.S. Army (Signal Corps), the U.S. Navy (Office of Naval Research) and the U.S. Air Force (Office of Scientific Research, Air Research and Development Command); and in part by the U.S. Atomic Energy Commission.

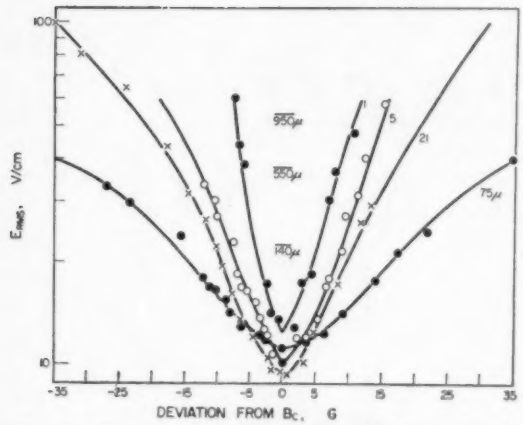


FIG. 2.—Plots of microwave breakdown field as a function of static magnetic field for various gas pressures. The short solid lines denote the breakdown fields at resonance for high gas pressures.

The breakdown curves are obtained as follows. The microwave frequency is tuned to the resonant frequency of the cavity with no discharge. From the amount of microwave power absorbed by the cavity and the Q-value of the cavity, the microwave electric field strength in the plasma region is computed. The power input is then increased for various values of ω_b until breakdown is obtained.

The interesting feature of the breakdown curves in Fig. 2 is that their half-widths decrease with pressure, until a pressure is reached for which the electron mean free path is of the order of the length of the tube and then the half-widths remain approximately constant (Fig. 3). It should be noted that only the length of tube need be considered since at these low pressures, the magnetic field renders the effective radius of the tube, as far as diffusion is concerned, much larger than its length. Similarly, the field at resonance necessary for breakdown decreases with pressure until the pressure is reached for which the mean free path

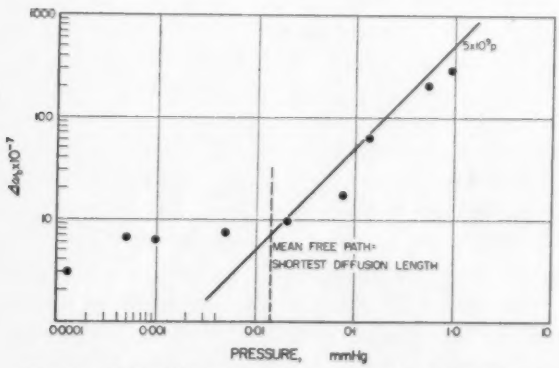


FIG. 3.—Half-widths of breakdown curves as a function of gas pressure. The constant half-width at pressures below 10 microns results partly from collisions with the end walls, and partly from the inhomogeneity of the magnetic field.

is equal to the tube length and then it increases slightly with decreasing pressures. Note that breakdown could be obtained for pressures as low as 0·1 microns when the average electron mean free path is some 20 times the length of the tube. This implies that an electron, on striking the end wall of the tube, is not necessarily lost, but is returned by some mechanism to the plasma. The nature of this mechanism will be exhibited later.

Once the gas is broken down, a plasma can be maintained, for a given input power, at pressures much smaller than 0·1 microns. For small input powers a plot of the pressure at which the plasma is *just* extinguished vs. the magnetic field, exhibits a minimum at cyclotron resonance. But, as the input power is increased (to hundreds of milliwatts) a similar plot of pressure against magnetic field is apt to be as shown in Fig. 4. Here, there is a peak at cyclotron resonance at a pressure p_1 with two minima,

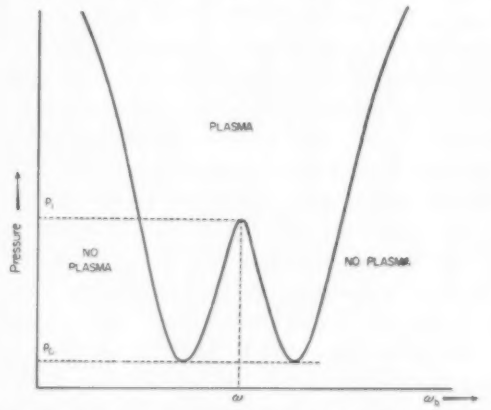


FIG. 4.—Pressure at extinction against magnetic field for a constant (large) input power.

at a pressure p_0 approximately equally spaced on each side of the resonance condition. The separation between the minima increases as the input power is increased.

The reason for this seemingly curious behaviour lies in the large energies which the electrons possess near cyclotron resonance (50–100 eV) with the resultant decrease in the relative ionization efficiency as compared with the electron loss rate, as the energy increases. The plot of Fig. 4 is symmetrical about the resonance condition because the electron energy is a function of $|\omega - \omega_b|$.* For more complete understanding of the rôle played by the electron energy we must inquire into the details of plasma production and

* From the equation of motion $m\,d\mathbf{v}/dt = e(\mathbf{E}\cos\omega t + \mathbf{v}\times\mathbf{B})$ it is easily determined that for an electron entering the field at $t = 0$ with zero initial velocity its energy at time t is given by

$$U = \frac{e^2E^2}{2m}\,\frac{\sin^2[\frac{1}{2}(\omega-\omega_b)t]}{(\omega-\omega_b)^2}.$$

maintenance. The microwave field increases the average electron energy to some value which enables the electron to ionize at a rate ν_i per second. This production rate is balanced by an average loss rate per electron, ν_L. Now, ν_i depends on the strength of the microwave field in the cavity, on the static magnetic field B_0, and on the electron collision frequency for momentum transfer ν_e, while ν_L depends on B_0, ν_e, the tube geometry and, as we shall see, it may depend rather critically on the plasma conditions near the boundaries. The relation $\nu_i = \nu_L$ defines a steady-state condition (such as breakdown or extinction). In hydrogen, the functional dependence of ν_i on electron energy is well-known (COMPTON, VAN VOORHIS, 1926). The plot of ν_i against electron energy for various gas pressures is shown in Fig. 5. Less is known about ν_L. At the low pressures and high magnetic fields at which these experiments are performed, electrons collide more often with the end walls of the quartz tube than with other plasma constituents. Consequently, ν_L should be independent of the neutral gas pressure. Let us postulate for the moment that it increases with electron energy as shown in Fig. 5 and justify this later. A number of features of the low density microwave plasma are evident from Fig. 5. The ν_L curve is tangent to some ν_i curve. This defines a pressure below which no plasma can be maintained irrespective of electron energy. At this lowest pressure, the electrons must have an energy U_{opt}, defined by the point of tangency, for the plasma to be maintained. The experimental plots of Fig. 6 are then a direct consequence of the physical situation

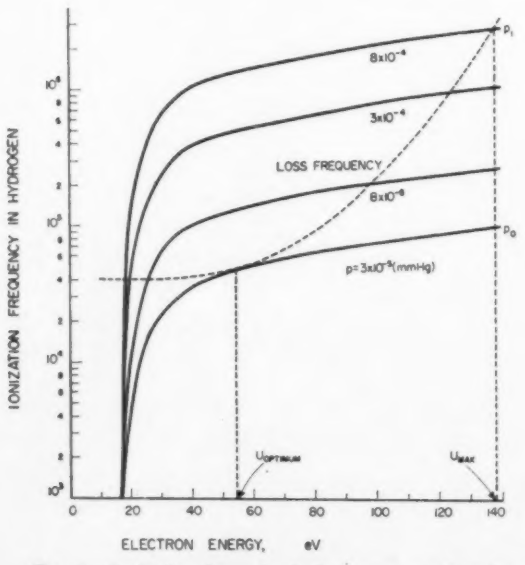


FIG. 5.—Ionization frequency in hydrogen as a function of electron energy and a postulated loss frequency.

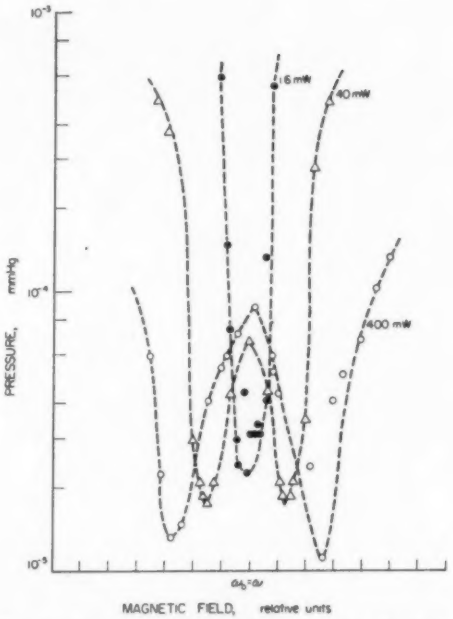


FIG. 6.—Experimental plots of pressure at extinction against the magnetic field for various input powers.

represented by Fig. 5. At low input powers, when the average electron energy at cyclotron resonance is less than U_{opt}, the experimental curves exhibit a single cyclotron resonance. At large input powers, a resonance obtains for those values of $|\omega - \omega_b|$ for which the electron energy corresponds to U_{opt}. As the input power is increased the separation between the minima increases as predicted. The experimental curves do not all reach the same minimum pressure. This is probably the result of differences in energy distributions of the electrons for different incident power levels and their effect on ν_L.

As a consequence of these experiments it may be concluded that near cyclotron resonance the electron energy is large (50 eV or more), in fact, sufficiently large for electrons to emit secondaries on striking the end walls of the quartz tube. If the yield of secondaries is less than unity the wall charges up negatively with respect to the plasma repelling all but the fastest electrons and attracting ions. Under this condition the loss rate ν_L is governed mainly by the drift velocity of the ions and is thus independent of electron energy. If the yield of secondaries is greater than unity, the end wall will ultimately charge up positively with respect to the plasma, reducing the effective yield to unity and repelling from the end wall the low velocity ions. Under these conditions the loss of particles from the plasma to the end walls is limited to only those fast ions which manage to climb the potential barrier. However, here there can also be loss to the

radial walls because the guiding centre of those secondary electrons which are returned to the plasma is displaced, on the average, by a Larmor radius from the guiding centre of the primary electron. This process enables the electrons to 'diffuse' across the magnetic field and reach the radial walls where recombination with ions can take place. The rate at which this diffusion takes place is proportional to electron energy. It is for these reasons that in Fig. 5, v_L is plotted as independent of electron energy for energies less than 30 eV (yield < 1) and increasing with energy for energies larger than 30 eV. That the 'diffusion' is the dominant loss mechanism at large energies is evidenced by the fact that as the diameter of the tube is increased, the minimum pressure at which the discharge can be maintained decreases.

Experiments were also performed in helium with similar results.

(b) *High-density plasma*

By a high-density plasma we mean one whose density is sufficient to perturb appreciably the microwave field configuration. Needless to say, this complicates the experiments considerably. The conditions for obtaining a high-density plasma are as follows. The neutral gas pressure is set anywhere between 1 and 5 microns. The cavity is tightly coupled to the microwave line, the microwave frequency is set equal to the resonant frequency of the empty cavity, and the gas is broken down by using cyclotron resonance described earlier. The amount of power required to accomplish this is small (1–10 mW). As the input power is increased the electron density increases. The first effect of the increasing density is a shift of the resonant frequency of the cavity. When the magnetic field is such that $\omega_b < \omega$, the shift is toward higher frequencies and vice versa. The magnetron is then tuned towards the new resonant frequency of the cavity in such a way that the microwave frequency is always between the empty cavity resonance and the actual resonant frequency. This assures operation on the stable part of the load characteristic. As long as the input power is modest (hundreds of milliwatts) the plasma is not yet dense enough to perturb the microwave field configuration (although it does shift the resonant frequency of the cavity) and the characteristics of the whole resonant structure are symmetric with respect to $|\omega - \omega_b|$.

As the density is increased so that the plasma frequency, ω_p, is such that $\omega_p^2 \simeq \omega^2 - \omega_b^2$, the microwave field configuration in the plasma is severely affected by the plasma when $\omega_b < \omega$. This is so because within the pseudo-static approximation (plasma radius much less than the free space wavelength) the plasma column resonates when $\omega_p^2/\omega^2 = 2(1 - \omega_b/\omega)$, and when this condition is satisfied the field in the plasma column is very large (in the absence of damping and density gradients it is infinite). Indeed, it is observed that with large input powers (20–50) watts decreasing the magnetic field increases the plasma density.

An interesting phenomenon takes place, in connexion with this resonance condition, when the input power is only of the order of 1 watt or so. Here it is observed that as ω_b is decreased from ω, the plasma column constricts to a filament, which can be made as narrow as 1 mm. With other parameters held constant, the radius of the constricted plasma column decreases with ω_b in such a way that the total number of particles in the plasma column remains constant. The detailed mechanism of this constricted discharge is not yet fully understood.

It may appear from the foregoing that the highest densities which we can obtain are limited by the frequency of the magnetron (i.e. $\omega_p^2 < \omega^2$). This is not the case on two accounts. Firstly, since the electron density is not uniform in the radial direction the resonant condition $\omega_p^2/\omega^2 = 2(1 - \omega_b/\omega)$ is first met at the outer edge of the plasma, so that the densities on the axis can be considerably higher. This is observed. As ω_b is decreased from ω, at large input powers (30–50 watts), the plasma column appears to be hollow in that most of the light is emitted from the outer edge of the plasma, while the measurement of the total number of electrons in the plasma indicates an average electron density in excess of that given by $\omega_p^2/\omega^2 = 2(1 - \omega_b/\omega)$.

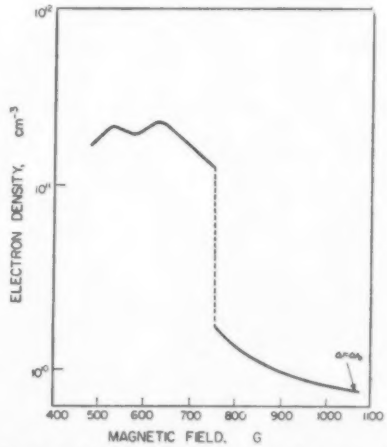


FIG. 7.—A typical experimental plot of measured average electron density against magnetic field. The cavity is initially overcoupled by approximately 30 dB. Input power is 40 watts. Microwave frequency is 50 Mc/sec higher than the resonant frequency of cavity with no plasma. Gas pressure is 2 microns.

Secondly, there is an additional effect which considerably increases the plasma density. As the parameter ω_b/ω is decreased toward the value 0·5, there occurs a discontinuous jump in the measured electron density and in the light output (see Fig. 7). On decreasing ω_b/ω further there is an additional increase in density, then a shallow minimum at approximately $\omega_b/\omega = 0{\cdot}5$, and then a sudden extinction of the plasma, usually accompanied by large oscillations in the megacycle region. The details of this high-density behaviour are not fully understood, but it is under these conditions that we achieve the largest degrees of ionization. In steady state, at an expenditure of 50 watts, approximately 2 per cent ionization has been obtained, while under pulsed conditions, when more peak power is available, over 10 per cent was achieved.

Acknowledgements—The authors wish to thank Mr. H. FIELDS who obtained the curves in Figs. 2 and 3 and Dr. D. O. AKHURST who helped obtain the curves in Fig. 6. Many helpful discussions were held with Drs. D. R. WHITEHOUSE and G. BEKEFI. The technical assistance of Mr. J. J. MCCARTHY is appreciated.

REFERENCES

BUCHSBAUM S. J. (1958) *Microwave Measurements of Plasmas in Magnetic Fields*, Quarterly Progress Report, p. 10, Research Laboratory of Electronics, M.I.T.

BUCHBAUM S. J., MOWER L. and BROWN S. C. (1960) *Phys. Fluids* **3**. In press.

COMPTON K. T. and VOORHIS C. C. VAN (1926) *Phys. Rev.* **27**, 724.

VOL.
2
196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 169 to 172. Pergamon Press Ltd.

III. STATISTICAL AND THERMODYNAMICAL PLASMA PHYSICS

THE APPROACH TO EQUILIBRIUM OF A PLASMA

R. BALESCU
Université Libre de Bruxelles, Bruxelles, Belgium

Abstract—After a short presentation of the general theory of irreversible processes established by PRIGOGINE and BALESCU, we discuss in some detail its application to the problem of long range forces in a plasma. Our diagram technique permits an unambiguous choice of the main contributions to the distribution function. The result is an equation describing the evolution of the distribution function of momenta, which takes rigorously into account the many-body collisions. The latter results from a summation of an infinite number of diagrams, the structure of which is very simple. The successive diagrams involve more and more particles interacting simultaneously.

The equation has a formal similarity with the usual Fokker-Planck equation, but the Coulomb interaction between two particles is replaced by an effective interaction, which depends on the distribution of all the other particles of the medium. At very low temperatures this effective interaction reduces to a screened Coulomb-Debye potential.

The relation of our approach to the previous theories of plasmas (Fokker-Planck equation, Vlasov equation, random phase approximation) can be discussed very clearly by means of the diagrams.

WE present a method for treating irreversible processes in a plasma, which aims to take rigorously into account the effect of collective interactions. This work is a consequence of the general theory of irreversible phenomena developed by PRIGOGINE and co-workers, more especially by PRIGOGINE and BALESCU (1959). We shall therefore outline the main ideas of the general method before discussing its applications to plasmas.

The Hamiltonian of a classical gas can be written as:

$$H = H_0 + \lambda V \tag{1}$$

where H_0 is the kinetic energy of the molecules, λV their interaction energy and λ a parameter measuring the strength of the interactions (in a plasma it is essentially e^2, the square of the charge).

The mechanical evolution of the system characterized by this Hamiltonian can be described by the Liouville equation

$$\frac{\partial f_N}{\partial t} = [H, f_N] \tag{2}$$

where $f_N = f_N(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N, t)$ is the distribution function in the complete phase space and the right-hand side is the usual Poisson bracket. The Liouville equation is equivalent to the set of $6N$ equations of motion of the particles; however the more compact formulation (2) is a more convenient starting point for an approximate solution of the problem.

We shall consider in our work a class of physical systems which is characterized by the following property: in the limit of an infinite number of particles, and of an infinite volume Ω, such that:

$$\begin{gathered} N \to \infty \\ \Omega \to \infty \\ N/\Omega = c = \text{constant} \end{gathered} \tag{3}$$

all the reduced distribution functions of a finite number of particles remain finite. This property means that all the *intensive* properties of the system, such as the density, the local average velocity, the binary density correlations, etc., have a finite value in this limit. This class of systems covers most of the systems which can be realized in practice.

It can be shown that this property has the following consequence. One can expand f_N in a Fourier series in which each type of coefficient has a volume dependence determined by our requirement. Writing explicitly all Ω factors, this expansion has the following form:

$$\begin{aligned} f_N = \Omega^{-N}\{\rho_0 &+ \Omega^{-1} \sum_{j=1}^{N} \sum_{\mathbf{k}}{}' \rho_{\mathbf{k}}^{j} \exp(i\mathbf{k}\cdot\mathbf{x}_j) \\ &+ \Omega^{-2} \sum_{j,n} \sum_{\substack{\mathbf{k},\mathbf{k}' \\ (\mathbf{k}+\mathbf{k}'\pm 0)}}{}' \rho_{\mathbf{k},\mathbf{k}'}^{jn} \exp(i\mathbf{k}\cdot\mathbf{x}_j + i\mathbf{k}'\cdot\mathbf{x}_n) \\ &+ \Omega^{-1} \sum_{j,n} \sum_{\mathbf{k}}{}' \rho_{\mathbf{k},-\mathbf{k}}^{jn} \exp[i\,\mathbf{k}\cdot(\mathbf{x}_j - \mathbf{x}_n)] + \ldots\} \end{aligned} \tag{4}$$

where ρ_0, $\rho_{\mathbf{k}}^{j}$, ... are functions of the momenta and of the time, and are independent of N and Ω. They have simple physical meanings. ρ_0, the coefficient

corresponding to vanishing wave vectors for all the particles, is simply the momentum distribution function. The coefficient $\rho_{\mathbf{k}}^{j}$ corresponding to a single non-vanishing wave vector for molecule j, is the Fourier transform of the local density fluctuation; it characterizes the inhomogeneity of the system. $\rho_{\mathbf{k},\mathbf{k}'}^{jn}$ and $\rho_{\mathbf{k},-\mathbf{k}}^{jn}$ both describe binary correlations; however the first one can be shown to factorize into two factors $\rho_{\mathbf{k}}$, whereas the second one is an irreducible correlation. This kind of properties are discussed in detail in PRIGOGINE's book (PRIGOGINE, to be published). It is clear that the expansion (4) plays the same role for non-equilibrium situations as the cluster expansion in equilibrium theory.

Introducing this expansion into the Liouville equation (2) we will obtain a set of equations relating the change in time of any Fourier component $\rho_{\{\mathbf{k}\}}$ to the value of a set of other components $\rho_{\{\mathbf{k}'\}}$. The corresponding 'matrix elements' of the perturbed Liouville operator obey the following selection rules: a transition from the set $\{\mathbf{k}\}$ to the set $\{\mathbf{k}'\}$ is possible only if all the wave vectors of the two sets are identical, except for two particles j and n; and if moreover $\mathbf{k}_j + \mathbf{k}_n = \mathbf{k}_j' + \mathbf{k}_n'$. These selection rules are a direct consequence of the assumption that the interactions are binary and spherically symmetrical [$V = \sum V_{jn}(|\mathbf{x}_j - \mathbf{x}_n|)$].

We now introduce diagrams to represent the transitions caused by the perturbation. A state described by a Fourier coefficient $\rho_{\{\mathbf{k}\}}$ will be represented by a number of lines equal to the number of non-vanishing wave vectors $\mathbf{k}$. A vertex represents a matrix element of the interaction. We can then represent the Fourier transformed Liouville equations by means of diagrams, as in Fig. 1.

One can say that these equations express classical mechanics as a dynamics of correlations. For instance, in the second equation we have a term which builds up a binary correlation from f_0, one which expresses a transfer of correlations from particles l and n to particles j and n, one expressing the continuation of the correlation between j and n after the interaction,

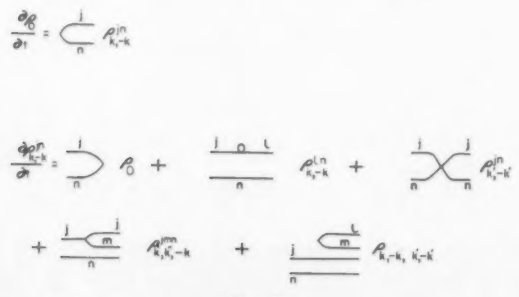


FIG. 1.

and two others expressing the decay of a ternary and a quaternary correlation to a double correlation.

We can now solve these equations by an iteration procedure. The contributions to any Fourier component will then be represented by diagrams having any number of vertices. The crucial point of the theory is the fact that one can establish theorems which determine the dependence of each diagram on the concentration c, on the coupling parameter λ and on the time t. These various dependences are related to simple topological properties of the diagrams, such as number of vertices, types of connexion of the vertices, diagonality, etc., which can be rapidly determined by a mere inspection. These theorems are proved in PRIGOGINE and BALESCU (1959); we shall not go into more details on this subject here, but rather consider some applications.

The simplest type of systems is the *weakly coupled gas*, in which λ is considered to be a small parameter. One can then show that for times which are long compared to the duration of a collision, the dominant terms of the the contributions to the reduced distribution function of momenta of particle α obey an

FIG. 2.

asymptotic equation which can be represented schematically as in Fig. 2. These contributions are proportional to arbitrary powers of the combination $\lambda^2 t$; any other contribution is proportional to $\lambda^n(\lambda^2 t)$ and is neglected in this approximation. The equation represented in Fig. 2 is a second-order differential equation in $\mathbf{p}$. It is well known that a plasma can be described in first approximation as a weakly coupled gas. If in (1) we suppose that λV is a Coulomb potential, then the equation of Fig. 2 has the following explicit form:

$$\frac{\partial \varphi(\alpha)}{\partial t} = e^4 c \int d\mathbf{l} \int d\mathbf{p}_1 \, \mathbf{l} \cdot \frac{\partial}{\partial \mathbf{p}_\alpha} \frac{\delta[\mathbf{l} \cdot (\mathbf{p}_\alpha - \mathbf{p}_1)/m]}{l^4} \times \mathbf{l} \left(\frac{\partial}{\partial \mathbf{p}_\alpha} - \frac{\partial}{\partial \mathbf{p}_1} \right) \varphi(\alpha)\varphi(1) \quad (5)$$

where $\varphi(\alpha) \equiv \varphi(\mathbf{p}_\alpha, t)$ is the distribution of momenta of particle α.

It can be shown (PRIGOGINE and BALESCU, 1957) that this equation is equivalent (although much more compact) to the well-known *Fokker-Planck equation*. Notice that the factor l^{-4} is essentially the square of the Fourier transform of the Coloumb potential. It is also well-known that the use of this equation leads to a divergence for long distances (small wave-vector

I). This divergence is due to the fact that equation (5) expresses a *binary collision* process, whereas the real interactions in a plasma are essentially *collective*, involving a large number of particles because of the long range of the forces.

A second type of approach, which has often been used in the theory of plasmas, is based on the idea of a *self-consistent field:* each particle is described as moving under the influence of an average field produced by the medium. The evolution of the distribution function of particle α in phase space, $f(\alpha) = f(\mathbf{x}_\alpha, \mathbf{p}_\alpha, t)$ is determined by an equation formally analogous to Liouville's equation for a free particle in a field $\mathbf{E}$:

$$\frac{\partial f(\alpha)}{\partial t} + \frac{\mathbf{p}_\alpha}{m} \cdot \frac{\partial f(\alpha)}{\partial \mathbf{x}_\alpha} + e\mathbf{E}(\mathbf{x}_\alpha, t) \cdot \frac{\partial \varphi(\alpha)}{\partial \mathbf{p}_\alpha} = 0, \qquad (6)$$

$\mathbf{E}(\mathbf{x}_\alpha, t)$ being due to the inhomogeneities of the distribution, and being defined through a Poisson equation:

$$(\partial/\partial \mathbf{x}_\alpha) \cdot \mathbf{E} = 4\pi e \int d\mathbf{p}_\alpha [f(\mathbf{x}_\alpha, \mathbf{p}_\alpha, t) - \varphi(\mathbf{p}_\alpha, t)]. \qquad (7)$$

This '*Vlasov equation*' cannot however be used for describing irreversible processes, because it contains no dissipative mechanism. Equation (6) can be obtained in our formalism, if we retain contributions proportional to any power of the combination $e^2\rho$. The equation for the Fourier coefficient $\rho_{\mathbf{k}}^{\alpha}$, will then be given by the diagram of Fig. 3. This diagram is very suggestive, as it shows that in this approximation the state of the system is entirely described by the Fourier coefficient $\rho_{\mathbf{k}}$, neglecting any interaction of different $\rho_{\mathbf{k}}$ and any higher order correlations. This is precisely the prescription of the BOHM and PINES *random phase approximation.*

The general theory of irreversible processes outlined previously enables us to treat the collective interactions in a plasma in a rigorous way. The important point is to remark that in this case neither e^2 nor the concentration c are significant expansion parameters. In equilibrium theory it has been rigorously shown (MAYER, 1950) that the dominant effect of the long-range interactions is to introduce an effective screened potential

$$V_{eff} = (e^2/r) \exp(-\kappa r), \qquad (8)$$

where the inverse Debye length κ is defined by:

$$\kappa^2 = 4\pi c e^2/kT. \qquad (9)$$

We thus see that the combination e^2c appears in the *exponential.* If we want to obtain a theory to the same degree of approximation as the Debye equilibrium theory, we have to retain in the expansions all powers of the combination e^2c.

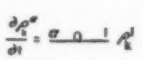


FIG. 3.

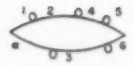


FIG. 4.

Consider now a homogeneous plasma (or rather, a homogeneous electron gas imbedded into a continuous neutralizing background) and examine the contributions to the momentum distribution function. It turns out that the dominant terms in the light of our preceding discussion are proportional to powers of the combination $e^2(e^2c)^n t$, $n > 0$. The corresponding diagrams are *rings* of the type shown in Fig. 4. Each ring describes a 'many-body collision process' involving particles α, 1, 2, ... The structure of these diagrams is very suggestive. In terms of correlations we see that during the collision process, two particles are correlated at each time, but these particles are not always the same, the correlation being successively transmitted from one particle to another.

Once we have chosen the significant diagrams, we can immediately show from the general theory that the asymptotic equation of evolution, valid for times which are long compared to a plasma oscillation period, is given by the scheme of Fig. 5. We shall not give here the details of the calculations, which will be found in another paper (BALESCU, 1960). Let us only mention that in a first step the summation is reduced to the solution of an integral equation, which can be solved exactly, leading to the following equation of evolution for the momentum distribution function:

$$\frac{\partial \varphi(\alpha)}{\partial t} = e^4 c \int d1 \int d\mathbf{p}_1 \, \mathbf{l} \cdot \frac{\partial}{\partial \mathbf{p}_\alpha} \times$$

$$\frac{\delta[\mathbf{l} \cdot (\mathbf{p}_\alpha - \mathbf{p}_1)/m]}{|l^2 + 4\pi i e^2 c \int d\mathbf{p}_2 \delta_-[\mathbf{l} \cdot (\mathbf{p}_\alpha - \mathbf{p}_2)/m] \mathbf{l} \cdot (\partial/\partial \mathbf{p}_2) \varphi(2)|^2}$$

$$\times \mathbf{l} \cdot \left(\frac{\partial}{\partial \mathbf{p}_\alpha} - \frac{\partial}{\partial \mathbf{p}_1} \right) \varphi(\alpha)\varphi(1). \qquad (10)$$

In order to understand the physical meaning of equation (10) it is important to remark that it has features in common both with the Fokker-Planck and with the Vlasov equation. This can already be seen by looking at the diagrams. The rings can be generated from the Fokker-Planck cycles, Fig. 2, by putting on the propagation lines an arbitrary number of Vlasov loops, Fig. 3. This means that the interaction between two electrons has to be renormalized in order to take into account the influence of the medium. In order

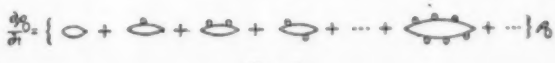


FIG. 5.

to see more precisely what this renormalization means consider an initially homogeneous plasma and a test-particle α moving with velocity $\mathbf{v}_\alpha$ through it. The plasma responds to this excitation by a polarization, i.e. by becoming inhomogeneous. We may calculate by Vlasov's equation the average field resulting from this polarization in a very simple way. It turns out (AKHIEZER and SITENKO, 1952) that the Fourier transform of this average field is:

$$\mathbf{E}_1 = \frac{e}{2\pi^2}\{l^2 + 4\pi i e^2 c \int d\mathbf{p}_1 \delta_-[\mathbf{l}\cdot(\mathbf{v}_\alpha - \mathbf{p}_1/m)] \times \mathbf{l}\cdot(\partial\varphi(1)/\partial\mathbf{p}_1)\}^{-1} i\mathbf{l}. \quad (11)$$

Comparing (10) with (5) we now clearly see the physical meaning of equation (10). *This equation results directly from the Fokker-Planck equation* (5) *if we replace in the latter the Coulomb interaction by the effective field calculated in the Vlasov approximation.* We may say that our equation describes collisions of 'dressed' electrons, thus taking into account the polarization of the medium. In this way the collective effects appear very naturally in the equation.

It is however important to stress that this analogy with the Fokker-Planck equation is purely formal, and that there is a great difference between the mathematical structures of equations (10) and (5). Physically, this is due to the fact that the polarization depends itself on the distribution function. This feature introduces a new type of non-linearity: besides the product of two distribution functions $\varphi(\alpha)\varphi(1)$—characteristic of any two-body collision term—equation (10) contains the distribution function in the *denominator*. The latter fact is specific of the many-body interactions, as can be seen by expanding the denominator: the equation then consists of a series of terms containing products of a larger and larger number of factors φ corresponding to the simultaneous interactions of an arbitrary number of particles.

Of course, the polarization does not even vanish in equilibrium. Thus, even in a problem of brownian motion (one particle out of equilibrium moving in a thermostat) equation (10) does not reduce to (5). One can show however that if the motion of the test-particle is slow with respect to the thermal velocity of the field particles expression (11) reduces to the Fourier transform of the Debye potential, equation (8)

$$\mathbf{E}_1^D = \frac{e}{2\pi^2}(l^2 + \kappa^2)^{-1} i\mathbf{l}. \quad (12)$$

This shows that equation (10) may reduce to a usual Fokker-Planck equation calculated with a Debye potential, but only under rather restrictive conditions. These restrictions are due to the fact that the polarization cloud has not in general the simple spherical form as in equilibrium, but is deformed by the motion of the particles.

Acknowledgements—We wish to express our sincere thanks to PROF. I. PRIGOGINE, who suggested this problem, for his constant interest and many illuminating discussions. We also wish to acknowledge some very interesting discussions with PROF. S. PRAGER from the University of Minnesota. This work has been done with the partial support of the Air Research and Development Command, USAF, through its European Office.

REFERENCES

AKHIEZER A. I. and SITENKO A. G. (1952) *J. Exp. Theor. Phys. U.S.S.R.* **23**, 161.

BALESCU R. (1960). *Phys. Fluids* **3**, 52.

MAYER J. (1950) *J. Chem. Phys.* **18**, 1426.

PRIGOGINE I. *Theory of Irreversible Processes*, Interscience, New York. To be published.

PRIGOGINE I. and BALESCU R. (1957) *Physica, s'Grav.* **23**, 555.

PRIGOGINE I. and BALESCU R. (1959) *Physica, s'Grav.* **25**, 281, 302.

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 173 to 183. Pergamon Press Ltd.

DYNAMICS OF CLASSICAL MANY-BODY SYSTEMS*

E. P. GROSS
Brandeis University, Waltham, Massachusetts

Abstract—In Part 1 we discuss the present state of the theory of the propagation of small amplitude disturbances in monatomic gases and plasmas. The analysis is based on kinetic equations for one-particle velocity distributions involving self-consistent field forces and Boltzmann-like collision terms. In Part 2 a microscopic formulation of the classical N-body problem is developed. The theory is used to analyse the meaning of earlier microscopic approaches, and carried to the point where the problem of Part 1 can be treated in a more fundamental way.

THE present contribution consists of two distinct but related parts. The first part deals primarily with the nature of the disturbances in a classical many-body system near thermal equilibrium. The problem is studied from the semi-phenomenological point of view provided by using a single-particle velocity distribution function. Attention is focused on the cases of monatomic neutral gases and plasmas. In the second part of our discussion, a very general microphysical approach to the classical N-body problem is presented. Emphasis is laid on the foundations and a separation between dynamical and statistical aspects is maintained, in order to understand more clearly the limitations of earlier work. The theory is developed to the point where an analysis of the disturbances about thermal equilibrium could be undertaken from a microscopic point of view, but we do not do that here.

1. PHENOMENOLOGICAL APPROACH—SINGLE-PARTICLE DISTRIBUTIONS

We look at the problem of describing the disturbances of a system near equilibrium from a phenomenological point of view. It appears possible to obtain a sensible description in terms of a single-particle velocity distribution in two limiting cases:

(1) low-density neutral monatomic gases where the mean free path is much longer than the molecular diameter (or range of interaction force),

(2) the low-density plasma where the mean free path for close collisions is much longer than the Debye length.

(1) *Neutral monatomic gases*

The theory of the dynamics of dilute neutral monatomic gases is considered to be in the best shape. Here the forces are short-ranged (compared to the mean distance between atoms) and binary collisions and bodily movement provide the dominant mechanisms for transfer of particles, momentum, energy, and other physical properties. The level of description is that of a one-particle distribution function $f(\mathbf{v}, \mathbf{x}, t)$, which represents the average number of particles in a volume element of a six-dimensional phase space $m\mathbf{v}, \mathbf{x}$. The averages are taken over space and velocity intervals, microscopic in size but large enough to contain a large number of particles. Sometimes averages over time intervals are taken. The use of a one-particle description means that certain types of microscopic fluctuations that occur in a real gas cannot be described. This is a limitation to which we will return later. From the distribution function one can compute macroscopic quantities such as the density $n(\mathbf{x}, t) \equiv \int f \, d\mathbf{v}$, the flow velocity, kinetic temperature, heat flow, stress, etc. The distribution function itself is taken to obey the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \frac{\mathbf{F}(\mathbf{v}, \mathbf{x}, t)}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\delta f}{\delta t}. \qquad (1)$$

The left-hand side describes the drift in phase space under the influence of an external body force $\mathbf{F}(\mathbf{v}, \mathbf{x}, t)$ while the right-hand side is constructed by considering binary collisions. The left-hand side of the equation preserves the exact reversibility of the Newtonian equations of motion. With $t \to -t$, $\mathbf{v} \to -\mathbf{v}$ and reversing external magnetic fields, the new distribution obey the same equation, if the right-hand side is zero.

Irreversibility is put explicitly into the construction of the right-hand side when one adopts the hypothesis of molecular chaos. In this hypothesis the particles which collide at a point $\mathbf{x}, t$ are assumed to be initially uncorrelated in velocity. The equations of motion of the interacting particles are used in describing the course of a collision and a collision results in a redistribution of particle velocities. The Boltzmann equation is irreversible, i.e. the substitution $t \to -t$,

* Work supported by Air Force, Cambridge, Mass.

$\mathbf{v} \to -\mathbf{v}$ no longer leads to a distribution function which satisfies the equation. Let us briefly summarize some of the properties of the Boltzmann collision term. It has the feature that the centres of the colliding molecules are assumed to be at the same positions, in the spirit of the assumption that the extent of the force field of an atom is small compared to the mean interatomic distance and the mean free path. For any velocity distribution $f(\mathbf{v}, \mathbf{x}, t)$, the quantities

$$\int \{1, \mathbf{v}, \mathbf{v}^2\} \frac{\delta f}{\delta t} d\mathbf{v}$$

vanish, expressing the conservation laws of particle number, momentum and energy during binary collisions at a given space-time point. The local Maxwellian distribution f_{eq} has the property $\delta f_{eq}/\delta t = 0$.

The simplest problems of the kinetic theory of gases are, in order of increasing difficulty: (1) thermal equilibrium; (2) the theory of steady-state transport; (3) the relaxation of a spatially homogeneous velocity distribution to the Maxwellian distribution; (4) the propagation of small-amplitude disturbances about equilibrium. Since (1) and (2) are well understood, we discuss (3) and (4).

The relaxation problem is the study of the integro-differential equation

$$\frac{\partial f}{\partial t} = \frac{\delta f}{\delta t}$$

when $f(\mathbf{v}, t = 0)$ is given. Of course, for any law of force, the entropy $k\int\int f \log f \, d\mathbf{v}$ increases monotonically with time. The details of the relaxation may be taken as an independent characterization of the force law for the binary interactions. For atoms repelling with an inverse fifth power force, Maxwell was able to find exact expressions for the decay of low velocity moments. The momentum and energy transport moments decay in a purely exponential manner with time, each with its own decay time. For this law of force the decay of all the velocity moments can be discussed (IKENBERRY and TRUESDELL, 1956). The solution of the relaxation problem can be expressed succinctly, if one restricts attention to the linearized Boltzmann equation. We introduce the equilibrium distribution

$$f_0 = n(\beta/\pi)^{1/2} \exp - \beta v^2$$

and write $f = f_0(1 + \phi)$, neglecting terms quadratic in ϕ. Then we have

$$\frac{\partial \phi}{\partial t} = k(\phi) = \int k(\mathbf{v}, \mathbf{v}')\phi(\mathbf{v}') \, d\mathbf{v}'. \tag{2}$$

k is a linear, symmetric operator and the discussion of the relaxation problem shifts to a determination of the eigenfunctions and eigenvalues of the operator. k is in fact the same operator that occurs in the Enskog-Chapman theory. For any law of force, the conservation laws insure that there is a five-fold degenerate zero eigenvalue. For central forces, the eigenvalues depend only on two proper values n, l and are degenerate with respect to an azimuthal proper value m. For 'inverse-fifth' molecules, WANG CHANG and UHLENBECK (1952) found the complete set of eigenvalues and eigenfunctions. The eigenvalues are

$$\lambda_{nl} = 2\pi \int_0^\pi d\vartheta \sin\vartheta \cos^{2n+l}(\vartheta/2) P_l(\cos\vartheta/2) \times [F(\vartheta) - F(\pi - \vartheta)] - A_0 - A_0 \delta_{n,0} \delta_{l,0} \tag{3}$$

where

$$\left.\begin{aligned} A_0 &= 2\pi \int_0^\pi F(\vartheta) \sin\vartheta \, d\vartheta, \\ gI(g\vartheta) &= \sqrt{\gamma/kI}\, F(\vartheta), \end{aligned}\right\} \tag{4}$$

γ is the force constant (i.e. γ/r^5), I is the scattering cross-section.

The eigenvalues are unbounded but increase very slowly as $(n + l + \text{const.})^{1/4}$. The inverse-fifth law is the only case where we know the eigenvalues and eigenfunctions. Thus it is the only case where the relaxation-time problem has been solved.

However, a knowledge of the properties of $k(\phi)$ does not lead directly to a solution of the initial-value problem for the propagation of a spatially inhomogeneous disturbance governed by the equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_x\right) \phi = k(\phi). \tag{5}$$

WANG CHANG and UHLENBECK (1952) and MOTT-SMITH have attempted to find approximate solutions of the form $\phi = \Sigma_\sigma a_\sigma(\mathbf{x}, t)\psi_\sigma(\mathbf{v})$, where $\psi_\sigma(\mathbf{v})$ are the eigenfunctions of k for inverse fifth molecules. This leads to an infinite set of partial differential equations for the a_σ which are characterized by the fact that the term $\mathbf{v} \cdot \nabla\phi$ couples a_σ to higher terms $a_{\sigma'}$ with $|\sigma'| > |\sigma|$. One has to adopt a scheme for terminating the equations, i.e. a moment procedure which arbitrarily expresses some velocity moment in terms of lower moments. Some success is achieved in describing disturbances whose space variation is small over a mean free path. But the relation between the drift effects (left-hand side of the Boltzmann equation) and the effects of collisions is obscured. In particular, it is hard to trace the evolution of initially sharp distributions in velocity space, since one needs to use many eigenfunctions to form a sharp perturbation in velocity space. The theory is inadequate to describe the propagation and absorption of waves whose length is of

VOL. 2 196

the order of or shorter than a mean free path. In fact, really effective use (other than to compute collision integrals) has not been made of the knowledge of the eigenfunctions and eigenvalues. To solve the initial-value problem for Maxwell molecules is an outstanding challenge in kinetic theory. This situation already warns us against over-optimism if we seek a complete theory of sound propagation in liquids and plasmas.

Nevertheless, a fairly detailed physical picture of the propagation of a disturbance can be obtained if one uses a kinetic model. A model involves replacing the collision term $\delta f/\delta t$ by a simpler mathematical expression. It is required that the model expression satisfy the conservation laws at each space-time point and lead to a monotonic entropy increase in the relaxation problem. The simplest model is defined by a single relaxation-time expression (BHATNAGER, GROSS and KROOK, 1954):

$$\frac{\delta f}{\delta t} = \frac{n}{\sigma}\{f_{eq} - f\} \tag{6}$$

where f_{eq} is the local Maxwellian distribution. The linear collision operator corresponding to this model is

$$k_M(\phi) = \frac{n_0}{\sigma}\left\{-\phi + v + \frac{m}{kT_0}\mathbf{v}\cdot\mathbf{q} + \tau\left(\frac{mv^2}{2kT_0} - \frac{3}{2}\right)\right\} \tag{7}$$

where

$$\left.\begin{aligned} v &= \int \phi F\, d\mathbf{v}, \quad \mathbf{q} = \int \mathbf{v}\phi F\, d\mathbf{v} \\ v + \tau &= \frac{m}{3kT_0}\int v^2 \phi F\, d\mathbf{v} \end{aligned}\right\} \tag{8}$$

and

$$F = \left(\frac{m}{2\pi kT_0}\right)^{3/2} \exp -\frac{mv^2}{2kT_0}.$$

This operator has the same eigenfunctions as does the collision operator for inverse-fifth molecules. It has a five-fold degenerate zero eigenvalue, expressing the conservation laws and one infinitely degenerate non-zero eigenvalue corresponding to the fact that it is a single relaxation-time model.

The important point is that $k_M(\phi)$ is a degenerate kernel, and an exact solution of the initial value problem is possible by the method of Laplace transforms. The solution is only slightly more complicated then LANDAU's result for the collision-free plasma. The main features of the solution are physically clear. In the time interval before the relaxation or mean collision time the distribution function evolves essentially as it does for a gas of non-interacting particles, i.e. $\phi(\mathbf{v}, \mathbf{x}, t) = \phi(\mathbf{v}, \mathbf{x} - \mathbf{v}t, 0)$. Velocity moments, such as the density or flow velocity, decay rapidly because of the different drift motions of particles. The precise manner of decay depends sensitively on the initial distribution of velocities. If the deviation from equilibrium is sharply peaked in velocity, the density decays very slowly. After the relaxation time the effects of the close collisions come into play. They cause a redistribution of velocities and thus arrest the drift 'damping' of macroscopic variables. The collisions tend to lock the particles to the local flow velocity and permit a spread in velocity which approximately defines a local temperature. From this point of view, we have the paradoxical situation that the irreversible terms in the Boltzmann equation stop the irreversibility inherent in drift damping. Because the collisions conserve energy and momentum, the Boltzmann equation leads to the reversible Euler equations of a perfect fluid describing the propagation of disturbances whose scale of length is large compared to a mean free path. The Euler equations are complete equations for the density and flow velocity together with a reversible adiabatic equation for temperature changes. They admit density waves which do not decay. It is only when we have a finite ratio of mean free path to wavelength that viscosity and heat conductivity effects come into play. As described by the Navier-Stokes equations and also the linearized kinetic equation, they lead to a *slow* decay of a macroscopic disturbance. As collisions become more frequent (shorter mean free path), a wave of a given length is less damped. These results at small and large values of time emerge from the solution based on the single relaxation-time model.

Now, one is of course concerned with the extent to which the description in terms of a simple kinetic model is trustworthy. The single relaxation-time model is certainly too simple, since there should be different relaxation times for the viscous stresses and energy flow. Thus the slow decay described by the Navier-Stokes equations is not correctly delineated. However, it is not hard to construct kinetic models with several relaxation times. Indeed, the Boltzmann operator can be decomposed into a model and a residual operator (GROSS and JACKSON, 1959). The improved models physically take into account the fact that particles are not scattered isotropically in the frame of reference defined by the local flow velocity. These models are mathematically more complicated than the single relaxation-time theory, yet the propagation of sound can still be treated exactly. They lead to the same overall description of the evolution of a disturbance. Furthermore, the residual operator can be analysed by perturbation methods and thus leads to an approximate solution of the Boltzmann equation itself.

The main defect of the models is that their eigenvalue spectra do not have the correct asymptotic dependence. Indeed they are bounded. This should affect the details of the evolution of the disturbance during the first collision interval. It would not be expected to destroy the qualitative picture of the interplay between drift damping and velocity-locking collisions. The higher eigenvalues describe, for example, the velocity dependence of the free path and persistence of velocities. It is possible to construct additional models which give an account of these features of any actual scattering potential. Of course, in many problems of kinetic theory, it is indispensable to consider some of these features and, if a model is to be useful, it must be chosen with proper attention to the situation. The simple models mentioned appear to be adequate to describe the main features of sound wave propagation. The understanding of their deficiencies leads to an appreciation of the complexity of this elementary phenomenon. At any rate, there exists at present no exact treatment of the propagation of disturbances on a basis provided by the linearized Boltzmann equation for some definite force law.

(2) *One-component plasma*

We shall first discuss the propagation of disturbances in a gas of interacting electrons moving on a smeared-out positively charged background to insure overall neutrality. On the phenomenological level implied by the use of a single-particle velocity distribution one generally introduces the self-consistent-field (SCF) interaction. To the external forces must be added a term

$$\left.\begin{aligned} \mathbf{F} &= -\nabla_x \int V(\mathbf{x}-\mathbf{x}')\{\rho(\mathbf{x}', t) - \rho_0\}\, d^3x', \\ \rho(\mathbf{x}', t) &= \int f(\mathbf{v}', \mathbf{x}', t)\, d^3v', \\ V(\mathbf{x}-\mathbf{x}') &= e^2/|\mathbf{x}-\mathbf{x}'|. \end{aligned}\right\} \quad (9)$$

When $\dfrac{\delta f}{\delta t}$ is set equal to zero, the kinetic equation is reversible. The complete linearized equation is

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\phi - \frac{1}{m}\frac{\partial f_0}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{x}} \times \left(\int V(\mathbf{x}-\mathbf{x}')\phi(\mathbf{v}', \mathbf{x}', t)\, d\mathbf{v}'\, d\mathbf{x}'\right) = K(\phi). \quad (10)$$

The self-consistent-field assumption makes no distinction between the effective field at a given particle and at an arbitrary point in space, i.e. it does not treat local field corrections. Actually the distribution of surrounding charges and hence the mean force acting on a given particle should depend on its velocity. This effect is not taken into account.

What are the new features that the self-consistent-field idea introduces? We first note that the SCF comes into play only when there is a density fluctuation $\rho(\mathbf{x}, t)$. Thus, if we study the pure relaxation problem, where we start initially with a distribution function independent of space, the net density at any point is, and remains, zero. Thus, we will have no interactions tending to cause a spread in velocity space unless there is density variation. In this sense, the self-consistent-field effect is unlike the binary collision mechanism. We might imagine that a modified self-consistent-field term which depends more directly on particle velocities might effect such a velocity spread. We have, however, as yet no elementary physical or mathematical argument leading to a definite form for such a modified SCF.

Nevertheless the usual SCF does have important effects. First, it makes possible collective density oscillations of sustained macroscopic amplitudes and with a frequency for very long wavelengths $\omega^2 \approx \dfrac{k^2 V_k}{m}$. For a plasma $V_k = 4\pi e^2/k^2$ and ω becomes the plasma frequency. For systems where $V_k \to V_0$ as $k \to 0$, we get a sound like-disturbance: $\omega \to k\sqrt{V_0/m}$. There are corrections to this formula arising from the thermal velocity spread. For a plasma, for long waves, $\omega^2 \approx \omega_p^2 + k^2\mathbf{v}^2$. The SCF interaction provides the organizing force for the collective motion. Nevertheless, the effects of the velocity spread and its tendency to produce decays of a macroscopic disturbance continue to show up. The collective motion exhibits a slow decay in time for long-wave vibrations and a rapid decay for shorter waves. (In the case of the plasma, the characteristic length is the Debye length $\lambda_D = (\sqrt{\mathbf{v}^2}/\omega_p)$. Physically this occurs because there is energy and momentum exchange between a collective wave and particles moving with a velocity approximately equal to the phase velocity of the wave. With a Maxwellian distribution for the particles there are more slow than fast particles, and there is a net loss of energy from the wave (LANDAU damping). The whole process has been analysed with precision by LANDAU (1946). His analysis is valid when it is legitimate to linearize the kinetic equation. Naturally, in thermal equilibrium, the density fluctuations are continually reappearing and decaying.

The similarity between the effects of the self-consistent field and of close collisions is striking. In both cases there is the initial drift damping because of a velocity spread; then there is the organizing influence of forces to create sustained macroscopic disturbances; finally there is a slow decay of a

disturbance because of its interaction with the individual particles. But the LANDAU result is based on a reversible equation, while the approach of the velocities to a Maxwellian distribution is inserted in an irreversible manner. This suggests that it may be possible to describe the diffusion in velocity space as a consequence of a time reversible equation. We will return to this point later.

Let us first summarize the consequences of combining the self-consistent field and a collision model. If one uses the single relaxation-time theory with a mean collision frequency $\lambda = n_0/\sigma$, the initial value problem is readily solved. The important parameter is the ratio of collision frequency to plasma frequency. Consider long waves such that $pa \ll \omega_p$ and $pa \ll \lambda$. p is the wave vector and $a = \left(\frac{kT_0}{m}\right)^{1/2}$ is the mean thermal velocity. The dispersion relation is found to be (BHATNAGER, GROSS and KROOK, 1954):

$$\omega \approx \omega_p + \frac{p^2a^2}{2\omega_p}\frac{1}{\lambda^2+\omega_p^2}\left\{\left(\frac{5}{3}\lambda^2 + 3\omega_p^2\right) - i\frac{4}{3}\lambda\omega_p\right\}. \tag{11}$$

For the low density plasma where $\lambda \ll \omega_p$, this reduces to the familiar result $\omega \to \omega_p + \frac{p^2a^2}{2\omega_p} \times 3$. The mathematical limit $\lambda \gg \omega_p$ represents the dominance of close collisions which produce hydrodynamic behaviour. One finds $\omega \to \omega_p + \frac{p^2a^2}{2\omega_p} \times \frac{5}{3}$. For the one-component plasma the term ω_p remains because any density variation implies electrical restoring forces. The coefficient 5/3 is characteristic of an adiabatic response of a system obeying the equation of state of a perfect gas. The system shows a peak in absorption at $\lambda = \omega_p$, as is expected when a transition is made from one type of wave propagation to another. This feature would probably remain in a more fundamental treatment of long wavelength disturbances in an electron gas. The behaviour at $\lambda \gg \omega_p$ is not correct since the electron gas would not obey the equation of state of a perfect gas. Nonetheless there is a transition from the self-consistent-field mode to a hydrodynamic mode. The corresponding theory for a two-component plasma where ionic motions are considered is of practical interest. One can understand qualitatively the conditions for the existence of plasma ion and electron oscillations, and trace the collision locking of ions and electrons to give rise to acoustic waves at high densities. For our purposes, however, it suffices to confine attention to the artificial electron gas model.

For the Coulomb case, collisions in which the impact parameter lies between a value e^2/kT (energy exchange $\sim kT$) and the Debye length $\lambda_D = (kT/4\pi ne^2)^{1/2} = \left(\frac{kT}{m}\right)^{1/2}\frac{1}{\omega_p}$ play the predominant role in causing velocity changes. These relatively soft collisions cause velocity changes due to the passing of two particles that are small compared to the original velocities. The process of thermalization is therefore visualized as a diffusion in velocity space and described by a Fokker-Planck equation. The simplest Fokker-Planck equation is defined by

$$\frac{\delta f}{\delta t} = D\frac{\partial}{\partial \mathbf{v}} \cdot \left\{\frac{kT}{m}\frac{\partial f}{\partial \mathbf{v}} + (\mathbf{v}-\mathbf{q})f\right\} \tag{12}$$

where D is a constant, $\mathbf{q} = \frac{\int \mathbf{v} f\, d\mathbf{v}}{\int f\, d\mathbf{v}}$, etc. It is noteworthy that we have introduced the velocity relative to the local flow velocity $\mathbf{q}(\mathbf{x}, t)$, and the local temperature T. One then sees that $\frac{\delta f}{\delta t}$ is compatible with all the conservation laws. Physically, the model represents an isotropic diffusion in velocity in a frame moving with a local velocity. This model is of course far simpler than the usual Fokker-Planck equation used in plasma theory. It is, however, quite different from the single relaxation time model. The eigenvalues of the linearized operators are linear in the proper values n and l. They thus rise asymptotically more rapidly than the inverse fifth eigenvalues. The elementary Fokker-Planck model has again the advantage that exact solutions of the sound propagation problem can be obtained. Although the problem has not been studied really thoroughly, it appears that the results are not appreciably different from those obtained with the relaxation time model (HAMMERLING, 1956). One can invent more sophisticated models which better simulate the important features of the accepted Fokker-Planck equation for plasmas. However, any theory involving a single-particle distribution function will be crude. In particular, such a theory has the unsatisfactory feature that in the hydrodynamic limit the dependence of the wave characteristics on the radial distribution function is not properly considered.

To treat the hydrodynamic limit, and also to obtain a theory of sound propagation in classical liquids, one must go to a more general formalism which involves multiple distribution functions. There exists the well-known chain of coupled equations for the multiple distribution functions. To start with, one may terminate the chain of equations by expressing, for example,

the triple correlations in terms of the double functions. From our point of view, the precise way of doing this is not initially important. It is important that the equations remain reversible in time. We can then ask: In what sense does the relaxation problem have meaning? and: Can we extend LANDAU's considerations to velocity diffusion? A start on these questions has been made by TCHEN (1959). It is clear from our review of the situation for the simpler Boltzmann equation that the solution is bound to be quite intricate. We shall not pursue the discussion of multiple distributions here, because the different but allied microscopic approach of Part 2 leads to the study of the same problems.

2. MICROSCOPIC THEORY

We will now develop a more fundamental microscopic theory of the classical N-body problem. The approaches which involve coarse-grained distribution functions pass to a statistical point of view at a very early stage. We will work with microscopic variables, i.e. with the analytical dynamics, to a larger extent before introducing statistical considerations. We will arrive at a theory that has been already presented by KLIMONTOVICH (1957) from a somewhat different point of view and by SCHONBERG (1952) in a more abstract form. The present line of thought was largely stimulated by conversations with BOHM in the summer of 1958.

Let us study N particles each of mass M interacting by central two-body potentials $V(r_{\alpha\beta})$. The Hamiltonian is then

$$H = \sum_{\alpha=1}^{N} \frac{P_\alpha^2}{2M} + \frac{1}{2} \sum_{\alpha \neq \beta} V(r_{\alpha\beta}). \tag{13}$$

For simplicity of presentation, let us consider potentials that can be decomposed in a Fourier series. We will work with periodic boundary conditions in a cube of unit volume. Then

$$H = \sum_{\alpha=1}^{N} \frac{p_\alpha^2}{2M} + \frac{1}{2} \sum_{\substack{\alpha \neq \beta \\ \mathbf{k}}} V_k \exp i\mathbf{k} \cdot (\mathbf{x}_\alpha - \mathbf{x}_\beta). \tag{14}$$

We find the equations of motion

$$M\dot{v}_\alpha{}^\mu = -i \sum_{\substack{\alpha \neq \beta \\ \mathbf{l}}} l^\mu V_l \exp i\mathbf{l} \cdot (\mathbf{x}_\alpha - \mathbf{x}_\beta),$$

$$\alpha, \beta = 1, \ldots, N, \qquad \mu = 1, 2, 3. \tag{15}$$

(We could include $\alpha = \beta$, since $\sum_{\mathbf{l}} l^\mu V_{\mathbf{l}} = 0$.)

The simplest dynamical variables of interest are symmetrical (and in fact additive) functions of the particle variables. Thus the density in space is

$$\rho(\mathbf{x}) = \sum_{\alpha=1}^{N} \delta(\mathbf{x} - \mathbf{x}_\alpha), \tag{16}$$

i.e. a sum of delta functions. Because the particles move, i.e. $\mathbf{x}_\alpha$ are functions of time, ρ also is a function of time. The microscopic density is a series of moving spikes, a discontinuous function in $\mathbf{x}$ space. Nevertheless, it evolves in a completely determined manner. Associated with this density are the Fourier components

$$\rho_{\mathbf{k}}(t) = \sum_{\alpha=1}^{N} \exp -i\mathbf{k} \cdot \mathbf{x}_\alpha. \tag{17}$$

Another such physical quantity is the microscopic current $j^\mu(\mathbf{x}, t) = \sum_{\alpha=1}^{N} v_\alpha{}^\mu \delta(\mathbf{x} - \mathbf{x}_\alpha)$, with the Fourier components $j_k{}^\mu(t) = \sum_{\alpha=1}^{N} v_\alpha{}^\mu \exp -i\mathbf{k} \cdot \mathbf{x}_\alpha$. These microscopic quantities are rigorously connected by the continuity equation

$$\frac{\partial \rho_k}{\partial t} = -ik^\mu j_k{}^\mu \quad \text{(summation convention)}. \tag{18}$$

It is however clear that other symmetrical sums of one-particle variables will be of interest e.g.

$$T_k = \frac{m}{2} \sum v_\alpha{}^2 \exp -i\mathbf{k} \cdot \mathbf{x}_\alpha . \tag{19}$$

In particular, this second velocity moment, for $k = 0$, is just the kinetic energy $\Sigma p_\alpha{}^2/2M$. The Hamiltonian is explicitly expressible in terms of these collective variables, i.e.

$$H = T_0 + \frac{1}{2} \sum_{\mathbf{k}} V_k \rho_k \rho_{-k}. \tag{20}$$

Even the binary interaction term is expressible in terms of sums of one-particle variables, at the price of being non-linear in the collective microscopic quantities.

Now, the continuity equation expresses the time change of ρ_k in terms of $j_k{}^\mu$. Let us, following the early work of BOHM and PINES (1952), write an equation for the time change of $j_k{}^\mu$. We have

$$\frac{\partial j_k{}^\mu}{\partial t} = \sum_{\alpha=1}^{N} \{v_\alpha{}^\mu(-i\mathbf{k} \cdot \mathbf{v}_\alpha) + \dot{v}_\alpha{}^\mu\} \exp -i\mathbf{k} \cdot \mathbf{x}_\alpha$$

$$= -i\mathbf{k} \cdot \sum \mathbf{v}_\alpha v_\alpha{}^\mu \exp -i\mathbf{k} \cdot \mathbf{x}_\alpha - i \sum l^\mu V_l \rho_{k-l} \rho_l. \tag{21}$$

When multiplied by the mass M, these are the equations of microscopic momentum balance. The first term on the right-hand side represents momentum flow per unit time. It involves second velocity moments. The second term represents momentum change per unit time arising from forces between particles.

It is non-linear in the collective variables. It is clear that, if we go on and form an expression for the time change of the second moments, two terms will be involved. One will be a higher velocity moment, the other a nonlinear function of collective variables. By proceeding in this way, we will find an infinite chain of equations.

Since all the symmetrical velocity moments are of physical interest, it would be desirable to consider the chain as a whole in a simple way. This can be done by introducing a generating function

$$G(\boldsymbol{\lambda}, \mathbf{k}, t) = \sum_{\alpha=1}^{N} \exp -i\boldsymbol{\lambda} \cdot \mathbf{v}_\alpha \exp -i\mathbf{k} \cdot \mathbf{x}_\alpha. \quad (22)$$

We have

$$\rho_{\mathbf{k}}(t) = G(0, \mathbf{k}, t), \quad j_k{}^{\mu}(t) = i \frac{\partial}{\partial \lambda_\mu} G(\boldsymbol{\lambda}, \mathbf{k}, t)|_{\lambda=0} \quad (23)$$

In fact a knowledge of $G(\boldsymbol{\lambda}, \mathbf{k}, t)$ yields all of the velocity moments by differentiations with respect to λ^μ. $G(\boldsymbol{\lambda}, \mathbf{k}, t)$ evolves in time as

$$\begin{aligned} \frac{\partial G}{\partial t}(\boldsymbol{\lambda}, \mathbf{k}, t) &= -i \sum_{\alpha=1}^{N} (\mathbf{k} \cdot \mathbf{v}_\alpha + \boldsymbol{\lambda} \cdot \mathbf{v}_\alpha) \\ &\quad \times \exp -i\boldsymbol{\lambda} \cdot \mathbf{v}_\alpha \exp -i\mathbf{k} \cdot \mathbf{x}_\alpha \\ &= \mathbf{k} \cdot \frac{\partial G}{\partial \boldsymbol{\lambda}} - \frac{\boldsymbol{\lambda}}{M} \sum_{\mathbf{l}} \mathbf{l} V_{\mathbf{l}} \rho_{\mathbf{l}} G(\boldsymbol{\lambda}, \mathbf{k} - \mathbf{l}, t), \end{aligned} \quad (24)$$

where we have used the equations of motion. Thus G satisfies a non-linear functional equation. It describes, in a rigorous, self-contained manner, the content of the many-body problem as projected on a space of symmetrical additive functions of the particle variables. The 'closure' with respect to the velocity moments is thus achieved simply. Indeed, G is closed under the Poisson bracket operation. Thus

$$\begin{aligned} \{G(\boldsymbol{\lambda}, \mathbf{k}), G(\boldsymbol{\lambda}', \mathbf{k}')\} &= \frac{1}{M}(\mathbf{k} \cdot \boldsymbol{\lambda}' - \mathbf{k}' \cdot \boldsymbol{\lambda}) \\ &\quad \times G(\boldsymbol{\lambda} + \boldsymbol{\lambda}', \mathbf{k} + \mathbf{k}'). \end{aligned} \quad (25)$$

The introduction of a generating function makes it possible to consider the whole chain of moment equations at once. We need not make arbitrary approximations which break the chain by expressing a higher moment in terms of lower moments. The treatment of the non-linear terms arises when we solve the equation for G. An approximation which linearizes the equation corresponds to a definite physical approximation, the neglect of close collisions. This is the random-phase approximation. We will discuss later how to include the effects of close collisions.

Before passing to the question of how to solve the equation (22) for $G(\boldsymbol{\lambda}, \mathbf{k}, t)$, we must discuss a number of points connected with the mathematical representation and the physical interpretation of the theory. First we note that the theory can be formulated in a number of equivalent ways. Equation (24) for $G(\boldsymbol{\lambda}, \mathbf{k}, t)$ is to provide the time-dependent behaviour of G for all values of G, $\boldsymbol{\lambda}$ and $\mathbf{k}$. If we Fourier transform both $\boldsymbol{\lambda}$ and $\mathbf{k}$ we go to velocity and position variables $\mathbf{v}$ and $\mathbf{x}$ (real space). In fact, the transform (we transform $\boldsymbol{\lambda}/M$ rather than $\boldsymbol{\lambda}$ so as to introduce the momentum) is:

$$N(\mathbf{p}, \mathbf{x}, t) = \sum_{\alpha=1}^{N} \delta(\mathbf{p} - \mathbf{p}_\alpha)\delta(\vec{x} - \vec{x}_\alpha). \quad (26)$$

We want to know $N(\mathbf{p}, \mathbf{x}, t)$ for all values of $\mathbf{p}$ and $\mathbf{x}$, i.e. in a 6-dimensional phase space. $N(t)$ is just the *fine-grained* density in the phase space. It is highly discontinuous. If one adds an extra dimension for purposes of visualization, $N(\mathbf{p}, \mathbf{x}, t)$ may be looked upon as a series of moving spikes coming out of the phase space. Alternative starting points are the mixed generating functions

$$E(\mathbf{p}, \mathbf{k}) = \sum_{\alpha=1}^{N} \delta(\mathbf{p} - \mathbf{p}_\alpha) \exp -i\mathbf{k} \cdot \mathbf{x}_\alpha,$$

$$F(\boldsymbol{\lambda}, \mathbf{x}) = \sum_{\alpha=1}^{N} \exp -i \frac{\boldsymbol{\lambda} \cdot \mathbf{p}_\alpha}{M} \delta(\mathbf{x} - \mathbf{x}_\alpha).$$

The equation that $N(\mathbf{p}, \mathbf{x}, t)$ satisfies can be found by Fourier transformation of the equation for $G(\boldsymbol{\lambda}, \mathbf{k}, t)$. Alternatively we may find it directly by forming

$$\begin{aligned} \frac{\partial}{\partial t} N(\mathbf{p}, \mathbf{x}, t) = \sum_{\alpha=1}^{N} &- \mathbf{v}_\alpha \cdot \frac{\partial}{\partial \mathbf{x}} (\delta(\mathbf{x} - \mathbf{x}_\alpha)\, \delta(\mathbf{p} - \mathbf{p}_\alpha)) \\ &- \dot{\mathbf{p}}_\alpha \cdot \frac{\partial}{\partial \mathbf{p}} (\delta(\mathbf{x} - \mathbf{x}_\alpha)\, \delta(\mathbf{p} - \mathbf{p}_\alpha)). \end{aligned}$$

Now the equations of motion are

$$\dot{\mathbf{x}}_\alpha = \frac{\mathbf{p}_\alpha}{M}, \qquad \dot{\mathbf{p}}_\alpha = -\frac{\partial}{\partial \mathbf{x}_\alpha} \sum_{\beta \neq \alpha} V(\mathbf{x}_\alpha - \mathbf{x}_\beta).$$

Making use of the properties of the δ-function, we can write

$$\begin{aligned} &\frac{\partial}{\partial t} N(\mathbf{p}, \mathbf{x}) + \frac{\mathbf{p}}{M} \cdot \frac{\partial N}{\partial \mathbf{x}} \\ &\quad = \sum_\alpha \frac{\partial}{\partial \mathbf{p}} \cdot \left\{ \frac{\partial}{\partial \mathbf{x}} \sum_\beta V(\mathbf{x} - \mathbf{x}_\beta) \right\} \\ &\qquad \times \delta(\mathbf{x} - \mathbf{x}_\alpha)\, \delta(\mathbf{p} - \mathbf{p}_\alpha) \\ &\quad = \frac{\partial N(\mathbf{p}, \mathbf{x})}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{x}} \sum_\beta V(\mathbf{x} - \mathbf{x}_\beta) \\ &\qquad \times \int \delta(\mathbf{x}'' - \mathbf{x}_\beta)\, \delta(\mathbf{p}'' - \mathbf{p}_\beta)\, d^3x''\, d^3p'' \\ &\quad = \frac{\partial N(\mathbf{p}, \mathbf{x})}{d\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{x}} \int V(\mathbf{x} - \mathbf{x}'') \\ &\qquad \times N(\mathbf{p}'', \mathbf{x}'')\, d^3p''\, d^3x''. \end{aligned} \quad (27)$$

This is the starting point of KLIMONTOVICH's theory. We have gone into the derivation in great deal because it was initially somewhat puzzling as to how he obtained the equation.

The remarkable thing is that $N(\mathbf{p}, \mathbf{x})$ obeys just the same self-consistent kinetic equation that is used for ordinary single-particle distribution functions $f(\mathbf{v}, \mathbf{x}, t)$, i.e. the Vlasov equation. Yet the considerations purport to be exact deductions from the equations of motion. How can this be true? We note that we have had to make two concessions to achieve this reduction. One is that we must operate with functions $N(\mathbf{p}, \mathbf{x})$ discontinuous in the phase space or with $G(\boldsymbol{\lambda}, \mathbf{k}, t)$ which for any $\boldsymbol{\lambda}$ and $\mathbf{k}$ are randomly fluctuating functions of time. The second concession is that we can only describe phenomena which depend on symmetrical functions of the particle co-ordinates. For the $N(\mathbf{p}, \mathbf{x})$ or $G(\boldsymbol{\lambda}, \mathbf{k}, t)$ are symmetrical and have the closure property, so that everything is built up from them. Thus we treat the content of the many-body problem insofar as it can be described in terms of symmetrical functions. At this point, the analogy to the treatment of the many-body problem in quantum-mechanics is illuminating. In quantum theory the Schrödinger equation is a linear wave equation for a function $\Psi'(\mathbf{x}_1, \ldots, \mathbf{x}_N, t)$, i.e. a function of $3N$-co-ordinates and time, with continuity properties. The equation permits general solutions in which Ψ' is neither a symmetrical nor an antisymmetrical function of the particle co-ordinates. The restriction to the one or the other type of function is an additional postulate. The restriction is, of course, compatible with the equations—for example, if a wave function is initially symmetrical, it remains so in the course of time. Now the method of second quantization makes possible a description in a three-dimensional rather than in a $3N$-dimensional co-ordinate or configuration space. To do this, one must first work with *operators* $\psi(\mathbf{x}, t)$, $\psi^+(\mathbf{x}, t)$, with singular commutation properties, the analogue of the discontinuous property of $N(\mathbf{p}, \mathbf{x}, t)$. Second, $\psi(\mathbf{x}, t)$ obeys a non-linear operator equation of the Hartree self-consistent-field type, i.e.

$$i\hbar \frac{\partial \psi}{\partial t}(\mathbf{x}, t) = -\frac{\hbar^2}{2M} \nabla^2 \psi + \psi^+ \times \int V(\mathbf{x} - \mathbf{x}') \psi^+(\mathbf{x}') \psi(\mathbf{x}')\, d^3x'.$$

With the interpretation of $\psi(\mathbf{x}, t)$ as an operator, the self-consistent-field equation is rigorous, whereas the Hartree equation for a product wave function $\Psi' = \Psi'_1(\mathbf{x}_1, t) \ldots \Psi'_N(\mathbf{x}_N, t)$ is only approximate. (Of course, in the quantum theory, the fact that we can only describe processes in which the wave function has a definite symmetry is a success of the theory). It is therefore clear that the title of KLIMONTOVICH's paper, *On the Method of Second Quantization in Phase Space*, is an apt one. We may note that some years ago SCHONBERG remarked that the procedure of second quantization has nothing to do with quantum mechanics, i.e. Planck's constant, *per se*. It is a method to redescribe the symmetrical (or antisymmetrical) solutions of a linear equation in many variables. He started from the similarity in this respect of Liouville's equation for a distribution function $f(\mathbf{p}_1 \ldots \mathbf{p}_N, \mathbf{x}_1 \ldots \mathbf{x}_N, t)$ and the Schrödinger equation in $3N$-configuration space. In a series of papers, he introduced operators $\psi(\mathbf{p}, \mathbf{x}, t)$ in a 6-dimensional phase space $\mathbf{p}$, $\mathbf{x}$ and developed a theory of second 'quantization' for classical dynamics. This is fully equivalent to, but more abstract than, the approach adopted here.

The purpose of these remarks has been in part to soften the initial shock of finding the self-consistent-field equation for the discontinuous functions $N(\mathbf{p}, \mathbf{x}, t)$. We must now turn to the physical interpretation of the formalism. This has been given by KLIMONTOVICH and is straightforward. The ordinary distribution functions which we wish to compare with experiment refer to coarse-grained quantities. One has to take averages over small regions of phase space or short-time averages. We do not want to enter into the difficult questions connected with the differences between types of averaging. Let us take phase space averages and represent them by bars. Then

$$f_1(\mathbf{p}, \mathbf{x}, t) = \overline{N(\mathbf{p}, \mathbf{x}, t)}, \tag{28}$$

where f_1 is the usual one-particle velocity distribution. The two-particle distribution is

$$f_2(\mathbf{p}, \mathbf{p}', \mathbf{x}, \mathbf{x}', t) = \overline{N(\mathbf{p}, \mathbf{x}, t) N(\mathbf{p}', \mathbf{x}', t)} - \delta(\mathbf{x} - \mathbf{x}')\, \delta(\mathbf{p} - \mathbf{p}') \overline{N(\mathbf{p}, \mathbf{x}, t)}. \tag{29}$$

The second term subtracts the self-interaction of particles. One can go on to define the complete hierarchy of multiple distributions in this way. We can check that the multiple distributions are connected in the usual way. This type of correspondence between coarse-grained and microscopic quantities has also been discussed by IRVING and KIRKWOOD (1950) and others. But we note that we can also compute correlations between phase-space densities at different times, as for example $\overline{N(\mathbf{p}, \mathbf{x}, t) N(\mathbf{p}', \mathbf{x}', t')}$ with $t' \neq t$.

The crucial point is that given some 'solution' of the microscopic equation, i.e. some $N(\mathbf{p}, \mathbf{x}, t)$, we can

compute the correlation functions of *all* order including those for different times. The questions 'What does it mean to 'solve' the microscopic equation?' and 'How far can we trust the answers which the recipes give for the subtler correlations?' are separate questions which must be taken up. What concerns us initially is that we have a *language* in terms of which we can express all the types of correlations.

Now let us proceed to the solution of the microscopic equations. We have the basic equation for $G(\boldsymbol{\lambda}, \mathbf{k}, t)$:

$$\left(\frac{\partial}{\partial t} - \mathbf{k}\cdot\frac{\partial}{\partial \boldsymbol{\lambda}}\right) G(\boldsymbol{\lambda}, \mathbf{k}, t) = -\frac{\boldsymbol{\lambda}}{M}\sum_{\mathbf{l}} \mathbf{l} V_{\mathbf{l}}\rho_{\mathbf{l}} \times G(\boldsymbol{\lambda}, \mathbf{k}-\mathbf{l}, t). \quad (30)$$

The first thing that suggests itself is to linearize the equations, i.e. make the random-phase approximation. Now $\rho_{\mathbf{l}}$ is a fluctuating quantity for $\mathbf{l} \neq 0$. If we take $\mathbf{l} = 0$, the right-hand side vanishes. Hence, to the first order in fluctuating quantities we must keep only $\mathbf{l} = \mathbf{k}$ on the right-hand side. Thus

$$\left(\frac{\partial}{\partial t} - \mathbf{k}\cdot\frac{\partial}{\partial \boldsymbol{\lambda}}\right) G(\boldsymbol{\lambda}, \mathbf{k}, t) \equiv -\frac{\boldsymbol{\lambda}\cdot\mathbf{k}}{M} V_{\mathbf{k}}\rho_{\mathbf{k}} G(\boldsymbol{\lambda}, 0, t). \quad (31)$$

If we are dealing with processes for which not only the deviation of the density is small but for which we are near thermal equilibrium, $G(\boldsymbol{\lambda}, 0, t)$ may be replaced by its average taken with the Gibbs' distribution, i.e.

$$\langle G(\boldsymbol{\lambda}, 0, t)\rangle = \left\langle \sum_{\alpha=1}^{N} \exp -i\boldsymbol{\lambda}\cdot\mathbf{v}_\alpha \right\rangle = N \exp \frac{\lambda^2 m}{4kT}. \quad (32)$$

Now the linearized equation for $G(\boldsymbol{\lambda}, \mathbf{k}, t)$ or for $N(\mathbf{p}, \mathbf{x}, t)$ is formally exactly the linearized Vlasov equation, and an initial value problem may be solved, for example, by LANDAU's Laplace transform method. We have to Fourier-transform $\boldsymbol{\lambda}$ (i.e. the most convenient generating function is $E(\mathbf{p}, \mathbf{k}, t)$) and Laplace-transform the time variable. If we assume that in some sense a solution is obtained, the solution is formally exactly the same as LANDAU's solution for the regular distribution $f(\boldsymbol{\lambda}, \mathbf{x}, t)$. But because of the different interpretation of $N(\mathbf{p}, \mathbf{x}, t)$, we may find expressions for the correlations of all order. This procedure seems to be the substance of the considerations of BOHM and PINES. Indeed, in the linearized theory the collective variables $\xi_k = \sum_{\alpha=1}^{N} \frac{\exp -i\mathbf{k}\cdot\mathbf{x}_\alpha}{\omega_k - \mathbf{k}, \mathbf{v}_\alpha}$ are—with a proper treatment of the singularity—closely related to the solution $G(\boldsymbol{\lambda}, \mathbf{k}, t)$. In the present scheme, the distinction between the dynamical and statistical is put in a sharper manner and thus has the virtue that it can be challenged and criticized. In addition, all the velocity moments are handled cleanly. The Russian work has been developed essentially to this point. If we take the above point of view, we may re-examine any problem for which the distribution function has been obtained from the self-consistent-field point of view, e.g. the small oscillations of a plasma in a magnetic field. The solution $N(\mathbf{p}, \mathbf{x}, t)$ can be used to calculate values for the correlations. The generalization to multicomponent systems is immediate. When one includes the full set of Maxwell's equations, the vector potential $\mathbf{A}(\mathbf{x}, t)$ is also a microscopically fluctuating quantity, so that a solution allows us to calculate correlations between the fluctuating field components at different space-time points.

The procedure of solving the microscopic equations can be given a more precise meaning in the following way. Starting with the approximate linearized microscopic equations, we can form equations for all of the coarse-grained multiple distributions. Thus we have (for single time correlations)

$$\frac{\partial f_1}{\partial t}(\mathbf{v}, \mathbf{x}, t) + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}} = -\frac{1}{M}\frac{\partial}{\partial \mathbf{x}}\int V(\mathbf{x}-\mathbf{x}') \times f_1\, d\mathbf{v}'\, d\mathbf{x}'\, \frac{\partial f_0}{\partial \mathbf{v}}, \quad (33)$$

$$\frac{\partial f_2}{\partial t} + \left(\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \mathbf{v}'\cdot\frac{\partial}{\partial \mathbf{x}'}\right) f_2(\mathbf{v}, \mathbf{v}', \mathbf{x}, \mathbf{x}', t) = \frac{1}{M}\frac{\partial f_0}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{x}}\int V(\mathbf{x}-\mathbf{x}'')\,\overline{NN''}\, d^3v''\, d^3x'' + \frac{1}{M}\frac{\partial f_0}{\partial \mathbf{V}'}\cdot\frac{\partial}{\partial \mathbf{x}'}\int V(\mathbf{x}'-\mathbf{x}'')\,\overline{N'N''}\, d^3v''\, d^3x''.$$

By comparison with the exact hierarchy, we see that we have

$$f_2(\mathbf{v}, \mathbf{v}', \mathbf{x}, \mathbf{x}', t) \to f_1(\mathbf{v}, \mathbf{x}, t)\, f_0(\mathbf{v}') + f_0(\mathbf{v})\, f_1(\mathbf{v}', \mathbf{x}', t),$$
$$f_3(\mathbf{v}, \mathbf{v}', \mathbf{v}'', \mathbf{x}, \mathbf{x}', \mathbf{x}'') \to f_2(\mathbf{v}, \mathbf{v}', \mathbf{x}, \mathbf{x}')\, f_0(\mathbf{v}'') + \text{cyclic} - \text{self interaction terms}.$$

The whole chain of equations is considered, and each multiple distribution function is approximated by lower functions. The approximation is of course closely tied to the treatment of disturbances near thermal equilibrium. Different methods of approximating the microscopic equations would be needed for other problems.

In this formalism the price of linearizing the equation for $G(\boldsymbol{\lambda}, \mathbf{k}, t)$ is that one does not give an account of close collisions. Now, one may take a harsh view of a procedure that approximates quantities such as $G(\boldsymbol{\lambda}, \mathbf{k}, t)G(\boldsymbol{\lambda}', \mathbf{k}', t)$ in the microscopic equations and

then goes on to compute the coarse-grained averages of the same quantities with the approximate solutions of the linearized microscopic quantities. It is, of course, true that one cannot trust the predictions for all space and time intervals. It is a question of the different characteristic lengths or times involved in the theory. The close collisions bring in a mean free path, which for low density plasmas, is long compared to the Debye length which is characteristic of the self-consistent field. Correlations of fluctuating charge might be accurate for distances between a Debye length and the mean free path. The theory might be faulty for distances between correlations which are less than a Debye length because the electric fields due to particular particles are large and rapidly fluctuating. The verdict that the procedure is devoid of meaning appears to be too harsh.

Nevertheless we are immediately led to the question of how to improve the theory. From the viewpoint presented here, this should come from 'solving' the microscopic equations more accurately, and then re-computing all the correlation functions.

It is important—as equation (21) shows—that the random-phase approximation leads to a violation of microscopic momentum and energy balance. Momentum and energy are lost by close collisions. To keep strict account, we must never break down the term containing a product of two collective variables.

To extend the theory, we introduce the two-particle correlation function

$$G_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{k}, \mathbf{k}') = \sum_{\alpha\beta}{}' \exp\left[-i\boldsymbol{\lambda}\cdot\mathbf{v}_\alpha - i\boldsymbol{\lambda}'\cdot\mathbf{v}_\beta - i\mathbf{k}\cdot\mathbf{x}_\alpha - i\mathbf{k}'\cdot\mathbf{x}_\beta\right]. \quad (34)$$

In fact,

$$G_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{k}, \mathbf{k}') = G(\boldsymbol{\lambda}, \mathbf{k}, t)G(\boldsymbol{\lambda}', \mathbf{k}', t), \quad (35)$$

in accordance with the fact that multiple correlations appear in the non-linear terms for the generating function. Actually we want to remove the term $\alpha = \beta$ which represents self-interactions. Therefore we define the 'irreducible' two-particle generating function

$$g_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{k}, \mathbf{k}') \equiv G(\boldsymbol{\lambda}, \mathbf{k})G(\boldsymbol{\lambda}', \mathbf{k}') - G(\boldsymbol{\lambda} + \boldsymbol{\lambda}' \mid \mathbf{k} + \mathbf{k}'). \quad (36)$$

Of course $g_1(\boldsymbol{\lambda}, \mathbf{k}) = G(\boldsymbol{\lambda}, \mathbf{k})$. In like manner we define $g_3(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \boldsymbol{\lambda}'' \mid \mathbf{k}, \mathbf{k}', \mathbf{k}'')$ by

$$\begin{aligned} G_3(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \boldsymbol{\lambda}'' \mid \mathbf{k}, \mathbf{k}', \mathbf{k}'') &\equiv G(\boldsymbol{\lambda}, \mathbf{k})G(\boldsymbol{\lambda}', \mathbf{k}')G(\boldsymbol{\lambda}'', \mathbf{k}'') \\ &= g_3(\boldsymbol{\lambda}, \boldsymbol{\lambda}', \boldsymbol{\lambda}'' \mid \mathbf{k}, \mathbf{k}', \mathbf{k}'') \\ &\quad + g_2(\boldsymbol{\lambda}' + \boldsymbol{\lambda}'', \boldsymbol{\lambda} \mid \mathbf{k}' + \mathbf{k}'', \mathbf{k}) \\ &\quad + g_2(\boldsymbol{\lambda} + \boldsymbol{\lambda}'', \boldsymbol{\lambda}' \mid \mathbf{k} + \mathbf{k}'', \mathbf{k}') \\ &\quad + g_2(\boldsymbol{\lambda} + \boldsymbol{\lambda}', \boldsymbol{\lambda}'' \mid \mathbf{k} + \mathbf{k}', \mathbf{k}'') \\ &\quad + g_1(\boldsymbol{\lambda} + \boldsymbol{\lambda}' + \boldsymbol{\lambda}'' \mid \mathbf{k} + \mathbf{k}' + \mathbf{k}''). \end{aligned} \quad (37)$$

We can now rewrite the fundamental equation for $G(\boldsymbol{\lambda}, \mathbf{k})$ as

$$\left(\frac{\partial}{\partial t} - \mathbf{k}\cdot\frac{\partial}{\partial \boldsymbol{\lambda}}\right) g_1(\boldsymbol{\lambda}, \mathbf{k}, t) = -\frac{\boldsymbol{\lambda}}{m}\sum_{\mathbf{l}} \mathbf{l}\, V_l\, g_2(0, \boldsymbol{\lambda} \mid \mathbf{l}, \mathbf{k} - \mathbf{l}) \quad (38a)$$

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \mathbf{k}\cdot\frac{\partial}{\partial \boldsymbol{\lambda}} - \mathbf{k}'\cdot\frac{\partial}{\partial \boldsymbol{\lambda}'}\right) g_2(\boldsymbol{\lambda}, \boldsymbol{\lambda} \mid \mathbf{k}, \mathbf{k}', t) \\ &= -\sum_{\mathbf{l}} \mathbf{l} V_l[\{\boldsymbol{\lambda} g_3(0, \boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{l}, \mathbf{k} - \mathbf{l}, \mathbf{k}') \\ &\quad + \text{interchange}\}] \\ &\quad + \{\boldsymbol{\lambda} g_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{k} + \mathbf{l}, \mathbf{k}' - \mathbf{l}) \\ &\quad + \text{interchange}\}, \end{aligned} \quad (38b)$$

where 'interchange' means $\boldsymbol{\lambda} \rightleftarrows \boldsymbol{\lambda}'$, $\mathbf{k} \rightleftarrows \mathbf{k}'$.

In this way we convert the original self-contained non-linear equation into a chain of equations. However we will now no longer tamper with equation (38a); this guarantees that the microscopic conservation laws are satisfied. Instead, we make an approximation in which g_3 is expressed in terms of g_1 and g_2. Since the approximation is made in the equations for microvariables, it is equivalent to an approximation for the entire chain of coarse-grained multiple distribution functions. We first note that $g_3(0, \boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{l}, \mathbf{k} - \mathbf{l}, \mathbf{k}')$ can be written in three alternative ways:

(a) $g_1(0, \mathbf{l})g_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{k} - \mathbf{l}, \mathbf{k}')$,

(b) $g_1(\boldsymbol{\lambda}, \mathbf{k} - \mathbf{l})g_2(0, \boldsymbol{\lambda}' \mid \mathbf{l}, \mathbf{k}')$,

(c) $g_1(\boldsymbol{\lambda}', \mathbf{k}')g_2(0, \boldsymbol{\lambda} \mid \mathbf{l}, \mathbf{k} - \mathbf{l})$.

Next we list those functions that have non-zero thermal averages (evaluated with the Gibbs' canonical distribution in $6N$-dimensional phase space). $g_1(\boldsymbol{\lambda}, \mathbf{k}, t)$ has a non-zero average only when $\mathbf{k} = 0$, i.e.

$$\langle g_1(\boldsymbol{\lambda}, \mathbf{k}, t)\rangle = \langle g_1(\boldsymbol{\lambda}, 0)\rangle\, \delta_{\mathbf{k},0}.$$

We also have

$$\langle g_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{k}, \mathbf{k}', t)\rangle = \langle g_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{k}, -\mathbf{k})\rangle\, \delta_{\mathbf{k}',-\mathbf{k}},$$

since $\langle g_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{k}, -\mathbf{k})$ is related to the k^{th} Fourier component of the two-body correlation function. The dependence on $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}'$ is of course elementary since the Gibbs' distribution factors in momentum space.

The linearized approximation of the terms (a), (b), (c) can now be written down. We put

$$g_1(0, \mathbf{l}) = \langle g_1(0, \mathbf{l})\rangle + \delta g_1(0, \mathbf{l}, t),$$

$$g_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{k}, \mathbf{k}', t) = \langle g_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}' \mid \mathbf{k}, \mathbf{k}')\rangle + \delta g_2,$$

and drop terms in which two factors involving δ appear. Equating terms which involve only thermal averages, we find expressions for the equilibrium correlation functions in the approximation considered. The quantities δg_1 and δg_2 then obey a pair of coupled linear integro-differential equations with constant coefficients that depend on the thermal averages. We thus have equations to treat the problem of small-amplitude wave propagation discussed in Part 1.

We will not write down these equations here, or enter into an analysis of them. This work is now being carried out in collaboration with Dr. C. Willis. It is clear from the discussion of Part 1 that explicit results can be looked for (1) in the long wavelength limit (moment methods), (2) by iteration of the integral equation, starting from the self-consistent-field limit, (3) by constructing model kernels and analysing their relations to the actual integral equations.

Acknowledgement—This discussion arrives at a formulation of the plasma problem in which the relation of microscopic randomly fluctuating quantities and observed averages stands in the forefront. It hardly touches the surface of this relation. It is unfortunate that DAVID BOHM was not able to be present at this symposium. For many years he has called attention to the fundamental nature of the theory of plasmas. In innumerable letters, conversations, and papers, he has emphasized the central position of the problem of the relation of random and regular processes—and developed many mathematical formalisms to deal with the relation. His influence is present both explicitly and in subtler ways.

REFERENCES

BHATNAGER P. L., GROSS E. P. and KROOK M. (1954) *Phys. Rev.* **94,** 511.
BOHM D. and PINES D. (1952) *Phys. Rev.* **85,** 338.
GROSS E. P. and JACKSON E. A. (1959) *Phys. Fluids* **2,** 432.
HAMMERLING P. (1956) Master's Thesis, Syracuse Univ.
IKENBERRY E. and TRUESDELL C. (1956) *J. Rat. Mech. Anal.* **5,** 1.
IRVING J. and KIRKWOOD J. (1950) *J. Chem. Phys.* **18,** 817.
KLIMONTOVICH Y. L. (1957) *Zh. eksp. teor. fiz.* **33,** 982.
LANDAU L. D. (1946) *J. Phys. USSR* **10.** 25.
SCHONBERG M. (1952) *Nuovo Cimento* **9,** 1139.
TCHEN C. M. (1959) *Phys. Rev.* **114,** 394.
WANG CHANG C. S. and UHLENBECK G. E. (1952) Univ. of Michigan Report.

OL.
2
961

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp 184 to 187. Pergamon Press Ltd. Printed in Northern Ireland

ON THE MOTION OF A CHARGED PARTICLE

I. PRIGOGINE
Faculté des Sciences de l'Université Libre de Bruxelles, Belgium

Abstract—The motion of a charged particle, including the effect of its self field is treated in the frame of classical mechanics by the Liouville method, developed recently by the author and his co-workers mainly for statistical mechanical problems. The author summarizes the results which he and LEAF has obtained in this way for the following cases:

(a) free particle in vacuum
(b) free particle in a black body
(c) harmonic oscillator.

In every case the partial differential equation giving the evolution of the particle phase distribution function is established.

1. INTRODUCTION

THE description of the classical motion of a charged particle or of a system of particles including radiation damping leads to well-known difficulties (LANDAU and LIFSHITZ, 1951). The usual procedure is based on the Lorentz equation:

$$m\mathbf{v} = \mathbf{F}_{\text{ext}} + \frac{2e^2}{3c^3}\ddot{\mathbf{v}}. \tag{1.1}$$

I would like to make first some remarks about the derivation of this equation and about its physical meaning. In the usual method the field equations are first solved formally in terms of retarded potentials. Then the potentials are introduced into the equations of motion. This gives the particle acceleration in terms of the motion of the same particle during a finite time interval. This equation is, therefore, very complicated and no exact solutions are known. The usual formal expansion in powers of r/c gives as its first term the Lorentz equation (1.1). It is generally agreed that (1.1) has meaning only as long as the radiation force is small compared to the external force. In this way, solutions corresponding to self-accelerated motion (such as for example, in the force-free case $\dot{\mathbf{v}} \sim \exp[3mc^3t/2e^2]$) are avoided. However, such restrictions have to be introduced *a posteriori*. Furthermore, there is some doubt about the extension of (1.1) to situations in which even higher time derivatives begin to play a role. For these reasons it is interesting to give an alternative discussion in which the field and the particle are treated in a more symmetrical way. We have found it convenient to start with the Liouville equation which expresses the evolution of the density in phase space for the whole system. This starting for a purely mechanical problem may seem at first somewhat surprising. But the advantage of this equation is to summarize the entire mechanical behaviour of the system in a simple linear equation. On the other hand, consideration of the set of Hamiltonian equations while mathematically equivalent is a problem of much greater complexity. Use of the Hamilton-Jacobi formalism seems out of the question, because of its non-linear character. Basically, the reason for applying the Liouville formalism to this problem is the same as for the problems in statistical mechanics: the occurrence of a very large number of degrees of freedom.

There is also another aspect, which I would like to stress. *Perhaps the most interesting feature of the Lorentz equation* (1.1) *is its non-Hamiltonian character due to the appearance of the higher time derivatives.* This non-Hamiltonian character is exactly what one has to expect as a result of the elimination of the field variables. However, the way in which this appears in equation (1.1) is rather curious. Indeed the appearance of v indicates that new conditions have to be imposed to select the right *solution*. For example, one might impose the supplementary condition that for t going to infinity the acceleration vanishes (DIRAC, 1938).

In our Liouville formalism, the non-Hamiltonian character appears in a different way. Indeed, let us call H the Hamiltonian of the field plus particle system and ρ the density in phase space. We then have the Liouville equation

$$\frac{\partial \rho}{\partial t} = [H, \rho]. \tag{1.2}$$

We shall write this in the form (BROUT and PRIGOGINE, 1956; PRIGOGINE and BALESCU, 1959):

$$i\frac{\partial \rho}{\partial t} = L\rho \tag{1.3}$$

where L is the 'Liouville operator' which is a Hermitian operator in phase space.

In this formulation the state of the system is expressed by ρ and its change is given by the Liouville operator. By integration of all field variables we obtain the phase density for the particle which we shall call ρ_{pa}. The question now is: what is the equation for the evolution of ρ_{pa}? There are clearly two cases which may occur: either we obtain an equation of the form

$$\frac{\partial \rho_{pa}}{\partial t} = [H_{pa}^{*}, \rho_{pa}] \tag{1.4}$$

(in this case, we have a Hamiltonian description with, if necessary, a modified (renormalized) Hamiltonian), or we obtain an equation of the form:

$$i\frac{\partial \rho_{pa}}{\partial t} = L_{pa}\,\rho_{pa} \tag{1.5}$$

where L need not to be a first order differential operator like in formula (1.3), and which cannot be reduced to the Hamiltonian form (1.4). We then have a non-Hamiltonian formula. But, even in this case we have a first-order equation in time and the specification of ρ_{pa} at a given time $t = 0$ determines its behaviour for all times. Before we go into more details about the physical implications of such an evolution equation, let us summarize briefly our method (PRIGOGINE and LEAF, 1959a).

2. THE LIOUVILLE METHOD

Let us consider a free particle in an electromagnetic field in the non-relativistic approximation. Other situations like, for example, a harmonic oscillator or relativistic corrections may be treated along the same lines. To first order in the charge the Hamiltonian may then be written in the form

$$H = H_0 + eV \tag{2.1}$$

where H_0 is the sum of the free particle Hamiltonian and the Hamiltonian due to the free field (the transversal part of the field).

$$H_0 = \frac{p^2}{2\mu} + \sum_{\lambda} (\nu_\lambda/2\pi)\,(J_\lambda + J_{-\lambda}) \tag{2.2}$$

The notation and the units are the same as used by HEITLER (1955). The free field contribution to each normal mode is expressed in action variables.* We will also use a Coulomb gauge. To the decomposition (2.1) of the Hamiltonian corresponds a decomposition of the Poisson bracket and therefore of the Liouville operator:

$$L = L_0 + e\,\delta L. \tag{2.3}$$

The operator L_0 has the simple form

$$L_0 = i\frac{\mathbf{p}}{\mu}\frac{\partial}{\partial \mathbf{q}} + i\sum_{\lambda}\frac{\nu_\lambda}{2\pi}\left(\frac{\partial}{\partial \omega_\lambda} + \frac{\partial}{\partial \omega_{-\lambda}}\right) \tag{2.4}$$

where $\boldsymbol{p}$ and $\boldsymbol{q}$ are the momentum and the co-ordinate of the particle and ω_λ the angle variable corresponding to the action variable J_λ. We may then expand the solution ρ of the 'perturbed' equation (1.3) in terms of the eigenfunctions of the unperturbed Liouville operator (2.4)

$$\rho = \sum_{\mathbf{k}(l_\lambda l_{-\lambda})} \rho_{\mathbf{k}(l_\lambda l_{-\lambda})} \times \exp i\{\mathbf{k}(\mathbf{q} - c\mathbf{p}t/\mu) - \sum_{\lambda} 2\pi[l_\lambda(\omega_\lambda - \nu_\lambda\, t/2\pi) + l_{-\lambda}(\omega_{-\lambda} - \nu_{-\lambda}\, t/2\pi)]\} \tag{2.5}$$

where $\mathbf{k} = (2\pi/\Omega^{1/3})\{n_1 n_2 n_3\}$

with $n_1, n_2, n_3, l_\lambda, l_{-\lambda}$

all integers, and the summation is over all sets of these integers; Ω is the volume.

Substituting in (1.3) we get

$$\begin{aligned}\partial \rho_{\bar{\mathbf{k}}\{l_\lambda l_{-\lambda}\}}/\partial t &= -e\exp\{i[\bar{\mathbf{k}}c\mathbf{p}/\mu - \sum_{\lambda}(\bar{l}_\lambda + \bar{l}_{-\lambda})\nu_\lambda]t \\ &\times \sum_{\mathbf{k}(l_\lambda, l_{-\lambda})} \langle \bar{\mathbf{k}}\{\bar{l}_\lambda \bar{l}_{-\lambda}\}\,|\delta L|\,\mathbf{k}\{l_\lambda l_{-\lambda}\}\rangle \\ &\times \exp\{-i[\mathbf{k}c\mathbf{p}/\mu - \sum_{\lambda}(l_\lambda + l_{-\lambda})\nu_\lambda]t\} \\ &\times \rho_{\mathbf{k}(l_\lambda l_{-\lambda})}\end{aligned} \tag{2.6}$$

The matrix element $\langle \bar{\mathbf{k}}\{\bar{l}_\lambda\, \bar{l}_{-\lambda}\}\,|\delta L|\,\mathbf{k}\{l_\lambda\, l_{-\lambda}\}\rangle$ is defined as

$$\begin{aligned}&\frac{1}{\Omega}\int\cdots\int d\mathbf{q}\prod_{\alpha} d\omega_\alpha\, d\omega_{-\alpha} \\ &\times \exp\{-i[\bar{\mathbf{k}}\cdot\mathbf{q} - 2\pi\sum_{\lambda}(\bar{l}_\lambda\omega_\lambda + \bar{l}_{-\lambda}\omega_{-\lambda})]\} \\ &\times \delta L\exp\{i\,[\mathbf{k}\cdot\mathbf{q} - 2\pi\sum_{\lambda}(l_\lambda\omega_\lambda + l_{-\lambda}\omega_{-\lambda})]\}\end{aligned} \tag{2.7}$$

It should be noticed that because of the definition of the Liouville operator, the matrix element (2.7) is still an operator acting on the action variables of the field and the momentum of the particle. It is easily shown, using the well known expression for the potential energy V in (2.1) in terms of the vector potential, that for a given set $\bar{\mathbf{k}}\{\bar{l}_\lambda \bar{l}_{-\lambda}\}$ all matrix

* It is in these variables that the unperturbed Liouville operator takes its simplest form.

elements connecting this set to $\mathbf{k}\{l_\lambda l_{-\lambda}\}$ vanish except those which satisfy the following conditions $\mathbf{k} = \bar{\mathbf{k}} \pm \mathbf{k}_\mu$, $l_\mu = \bar{l}_\mu \pm 1$, $l_\lambda = \bar{l}_\lambda$ for $\lambda \neq \mu$.

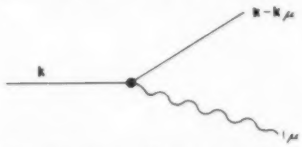


FIG. 1.—Diagram corresponding to a matrix element.

Let us represent by a line the particle and by a wavy line normal modes μ such that l_μ is different from zero. Each matrix element will then be represented by a diagram of the form given in Fig. 1. We shall be interested only in the behaviour of $\rho_{\mathbf{k}\{0,0\}}$. Indeed only these Fourier components determine the distribution function of the particle itself and therefore its motion. The equation (2.6) connects two Fourier coefficients like $\rho_{\mathbf{k}\{0,0\}}$ and $\rho_{\mathbf{k}-\mathbf{k}\mu,1\mu}$. The Fourier coefficients $\rho_{\mathbf{k}-\mathbf{k}\mu,1\mu}$ are in turn connected to $\rho_{\mathbf{k}\{0,0\}}$ and to other coefficients of the form $\rho_{\mathbf{k}'1\mu 1\mu'}$ (including the case where $\mu = \mu'$). In diagram form this is expressed in Fig. 2.

To order e^2 in the charge, the equation of motion for the particle corresponds to the diagram of Fig. 3. Note that the wavy line has a different meaning from that of the self-energy diagrams of quantum field theory. It represents a short-lived correlation between the particle and the normal mode involved and not the emission or the absorption of a photon. What we are indeed representing by a line would correspond in the quantum case to off-diagonal matrix elements of the density matrix.

We shall now discuss the explicit results derived from the diagram of Fig. 3, in some simple cases. The equations which we shall describe are valid always only for times long in respect to what one may call the duration of the correlation.

3. A FREE PARTICLE IN VACUUM

The simplest case corresponds to a free particle in vacuum. We define the vacuum by the following initial condition for the Fourier coefficients:

$$\rho_{\mathbf{k}\{0,0\}}(t=0) = \rho_{\mathbf{k}}(\mathbf{p}, t=0)\prod_\mu \delta(J_\mu/2)\,\delta(J_{-\mu}/2) \quad (3.1)$$

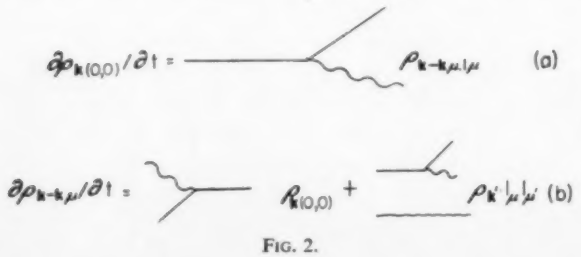


FIG. 2.

This condition implies that all action variables for the transversal part of the field have vanishing mean values and higher moments. We shall also use the notation

$$\rho_{\mathbf{k}}(\mathbf{p}, t) = \int \prod_\alpha dJ_\alpha\, dJ_{-\alpha}\rho_{\mathbf{k}\{0,0\}}(\mathbf{p}\{J_\mu J_{-\mu}\}, t) \quad (3.2)$$

The diagram of Fig. 3 then gives (PRIGOGINE and LEAF, 1959)

$$\partial\rho_{\mathbf{k}}/\partial t = i\omega_{\mathbf{k}}\rho_{\mathbf{k}} \quad (3.3)$$

where the frequency $\omega_{\mathbf{k}}$ is defined by

$$\omega_{\mathbf{k}} = \left(\frac{4\pi e^2 c}{\Omega\mu^2}\right)\sum_\mu k_\mu^{-2}\,[\mathbf{k}\cdot\mathbf{p} - (\mathbf{k}\cdot\mathbf{k}_\mu)(\mathbf{p}\cdot\mathbf{k}_\mu)/\mathbf{k}_\mu^2] \quad (3.4)$$

The only effect of the interaction of the particle with its electromagnetic field is that each Fourier coefficient ρ_k now oscillates in time with a frequency which

FIG. 3.

is both proportional to the momentum of the particle and to its wavelength. This oscillation expresses precisely the renormalization of mass. Indeed, if we use (3.3) and go back to phase space we obtain

$$\frac{\partial\rho_{p\alpha}}{\partial t} + \left[\frac{c\mathbf{p}_{p\alpha}}{\mu} + \frac{4\pi e^2 c}{\Omega\mu^2}\sum_\mu k_\mu^{-2} \times \left(\mathbf{p} - \frac{(\mathbf{p}\cdot\mathbf{k}_\mu)\mathbf{k}_\mu}{k_\mu^2}\right)\right]\frac{\partial\rho_{p\alpha}}{\partial\mathbf{q}} = 0 \quad (3.5)$$

which is the usual Liouville equation for a free particle with an effective mass, to order e^2

$$\mu_{\text{eff}} = \mu + \frac{4e^2}{3\pi}\int_0^\infty dk. \quad (3.6)$$

We, therefore, obtain an equation of the form (1.4). It should be noticed that we do not obtain a run-away solution. This may be connected to our method of expansion in powers of the charge.

4. FREE PARTICLE IN A BLACK BODY

Let us now consider a particle interacting with a black-body radiation field. Instead of (3.1) we now assume

$$\rho_{\mathbf{k}\{0,0\}}(t=0) = \rho_{\mathbf{k}}(\mathbf{p}, t=0)\prod_\mu \frac{\nu_\mu}{2\pi KT} \times \exp\left[-\gamma_\mu(J_\mu + J_{-\mu})/2\pi KT\right] \quad (4.1)$$

VOL 2 196

where K is Boltzmann's constant and T the absolute temperature. Repeating the same argument we now obtain

$$\frac{\partial \rho_{pa}}{\partial t} = -\frac{c\mathbf{p}}{\mu_{\text{eff}}}\frac{\partial \rho_{pa}}{\partial \mathbf{q}} - B\frac{\partial}{\partial \mathbf{p}}[p^2 I - \tfrac{1}{2}\mathbf{p}\cdot p]\frac{\partial}{\partial \mathbf{q}}\rho_{pa} \quad (4.2)$$

where μ_{eff} is the effective mass

$$\mu_{\text{eff}} = \mu + \frac{4e^2}{3\pi}\left(1 - \frac{7}{5}\frac{KT}{\mu}\right)\int_0^\infty dk \quad (4.3)$$

B is defined by

$$B = \frac{8e^2 cKT}{15\pi\mu^3}\int_0^\infty dk. \quad (4.4)$$

The electromagnetic self-mass is now temperature-dependent, a result which in principle permits an experimental determination of this term separately from the intrinsic particle mass μ. (4.2) is now a second order equation in the *canonical* variables pq. We now are in the situation (1.5). Clearly from (4.2) *canonical* equations of motion cannot be derived. The particle experiences a sort of *brownian* motion as it interacts with the oscillators of the electromagnetic field in thermal equilibrium. However, we still have for the average position and momentum.

$$\frac{d\bar{q}}{dt} = c\frac{\bar{p}}{\mu_{\text{eff}}} \quad (4.5)$$

$$\frac{d\bar{p}}{dt} = 0. \quad (4.6)$$

Consider the packet of points in the phase space given initially by the classical 'pure state'

$$\rho_{pa} = \delta(p - p_0)\,\delta(q - q_0 - c\mathbf{p}_0 t/\mu_{\text{eff}}) \text{ for } t = 0 \quad (4.7)$$

This expression would be a solution for all times in the case of the usual Hamiltonian formalism. But it is not a solution of (4.2) and accordingly, the initial packet would spread with time in the black-body case. For example, one may show that

$$\Delta\,\overline{\mathbf{qp}} = -\frac{e^2 B}{2}p_0^2 t. \quad (4.8)$$

The situation is strongly reminiscent of the Ehrenfest theorems in quantum mechanics.

5. HARMONIC OSCILLATOR

As our last example, let us consider a harmonic oscillator coupled to its electromagnetic field in the vacuum. The calculations are again the same (PRIGOGINE and LEAF, 1960b). The final equation for the particle distribution function is

$$\frac{\partial \rho_{pa}}{\partial t} + \frac{\nu}{2\pi}\frac{\partial \rho_{pa}}{\partial \omega} = \frac{1}{4\pi^2}\frac{e^2\nu^2}{\mu c}F\left[\left(\frac{J\nu}{2\pi\mu c}\right)^{1/2}\right]\frac{\partial \rho_{pa}}{\partial \omega} + \frac{2}{3}\frac{e^2\nu^2 J}{\mu c}\frac{\partial}{\partial J}\rho_{pa} + \frac{2}{3}\frac{e^2\nu^2}{\mu c}\rho_{pa} \quad (5.1)$$

where ω, J are the angle-action variables of the harmonic oscillator. The analytical form of the function F will not be given here. It corresponds to an (infinite) shift in frequency of the oscillation.

Again Equation (5.1) does not correspond to a Hamiltonian formalism, because of the last term of this equation. The equations for the average change of J and ω are easily deduced from (5.1). They are

$$d\bar{J}/dt = -2e^2\nu^2/3\mu c\bar{J} \quad (5.2)$$

$$d\bar{\omega}/dt = \nu/2\pi\left[1 - e^2\nu\,\overline{F\left(\left(\frac{J\nu}{\pi\mu c}\right)^{1/2}\right)}\Big/2\pi\mu c\right] \quad (5.3)$$

The first equation describes the radiation damping, correct to order e^2.

6. DISCUSSION

All the considerations we have developed can, of course, be put into a quantum mechanical formalism as well. One has to start with the fundamental equation for the statistical matrix and I expect that we shall come out with an equation for the evolution of the reduced density matrix for the particle which, however, in general can no more be reduced to a Schrödinger equation. I hope to develop this problem in a separate publication. I want only to mention that because of the Bose-Einstein distribution of the photons one has to expect a great reduction of the temperature dependence of the mass as given by (4.3) (PRIGOGINE and LEAF, 1959a).

Acknowledgements—I would like to thank very much DR. LEAF, who has performed most of the calculations, as well as DR. BALESCU for many interesting discussions.

REFERENCES

BROUT R. and PRIGOGINE I. (1956) *Physica, s'Grav.* **22**, 621.
DIRAC P. A. M. (1958) *Proc. Roy. Soc.*
HEITLER W. (1955) *Quantum Theory of Radiation* (3rd. Ed.) Oxford University Press.
LANDAU L. and LIFSHITZ E. (1951) *The Classical Theory of Fields*, Addison-Wesley.
PRIGOGINE. I and BALESCU R. (1959) *Physica, s'Grav.* **25**, 281, 302.
PRIGOGINE I. and LEAF B. (1959a) *Physica, s'Grav.* **25**, 1067.
PRIGOGINE I. and LEAF B. (1959b). Unpublished.

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 188 to 194. Pergamon Press Ltd. Printed in Northern Ireland

NON-EQUILIBRIUM THERMODYNAMICS OF SYSTEMS IN AN ELECTROMAGNETIC FIELD

S. R. DE GROOT
Lorentz Institute, University of Leyden, Leyden, Netherlands

Abstract—A review is given of some applications of thermodynamics of irreversible processes to systems in the presence of an electromagnetic field. The conservation laws of mass, momentum and energy and the entropy balance equation are developed for mixtures without and with electric and magnetic polarization, and also for systems in which the momentum transfer between some of their components is inhibited.

1. INTRODUCTION

MACROSCOPIC theory of irreversible processes can deal with a number of problems concerning irreversible processes in systems of which some components carry electrical charges. In the framework of thermodynamics of irreversible processes the conservation laws of mass, momentum and energy are set up for systems of which the properties are continuous functions of time and space co-ordinates. Then the entropy law is written down and with the help of the conservation laws an entropy balance equation is derived which contains a source term, the entropy production, which characterizes the irreversibility. This source term contains the fluxes and thermodynamic forces which are used in the description of irreversible behaviour (PRIGOGINE, 1947; DE GROOT, 1951; DE GROOT and MAZUR, to be published).

In the present review three particular cases are considered. In the first place (Section 2) systems are studied which consist of a mixture of chemical components, some of which carry electrical charges, but which are not supposed to be polarized by the electromagnetic field. In such systems heat conduction, diffusion, electric conduction, viscous flow and a number of cross-phenomena between these effects can take place. The thermodynamic theory of many of these effects has been developed. In the second place (Section 3) it is investigated how the scheme must be amplified if we also consider electric and magnetic polarization phenomena to be present in the system. We then have, besides the aforementioned effects, also electric and magnetic relaxation phenomena. Another problem concerning polarizable systems which can be treated in the framework of the theory is the question of pressure and ponderomotive force, which notions may be defined in different ways. Finally a third case (Section 4) is considered in which the transfer of momentum between some of the components of the system is hampered. The macroscopic theory of such a system rests on a more hypothetical basis than the theory of ordinary mixtures. One of the results is that a relaxation phenomenon of scalar heat flow between the components may arise. It is studied how this effect is coupled with a chemical reaction in the system.

2. SYSTEMS IN AN ELECTROMAGNETIC FIELD (WITHOUT POLARIZATION)

Let us consider a mixture of n components and write down the conservation laws of mass, momentum and energy as well as the entropy law and entropy balance equation.

(*a*) *Conservation of mass* is expressed by

$$\frac{\partial \rho_k}{\partial t} = -\text{div } \rho_k \mathbf{v}_k, \qquad (k = 1, 2, \ldots, n) \qquad (1)$$

where ρ_k is the density of component k, and $\mathbf{v}_k$ its velocity. We shall suppose in this section that no chemical reaction occurs; otherwise at the right-hand side of (1) one term more would arise, describing the production of substance k by chemical reactions (cf. however Section 4). An alternative form of (1) is

$$\rho \frac{dc_k}{dt} = -\text{div } \mathbf{J}_k, \qquad (2)$$

where $c_k = \rho_k/\rho$ (with $\rho = \Sigma_k \rho_k$) is mass fraction of k, and where

$$\mathbf{J}_k = \rho_k(\mathbf{v}_k - \mathbf{v}) \qquad (k = 1, 2, \ldots, n) \qquad (3)$$

represents the diffusion flow of k. The velocity $\mathbf{v}$ is the centre of gravity velocity defined by $\rho\mathbf{v} = \Sigma_k \rho_k \mathbf{v}_k$. The time derivative in (2) is the 'barycentric' substantial derivative $\partial/\partial t + \mathbf{v} \cdot \text{grad}$.

(*b*) *Conservation of momentum.* From Maxwell's equations for systems without electric and magnetic polarization ($\mathbf{E} = \mathbf{D}$, $\mathbf{H} = \mathbf{B}$) one can derive a balance equation of electromagnetic momentum of the following form

$$\frac{\partial \mathbf{g}}{\partial t} = \text{Div } T - \mathbf{F}, \qquad (4)$$

where $\mathbf{g}$ is the electromagnetic momentum density, T the Maxwell tensor of electromagnetic stresses and $-\mathbf{F}$ a source of momentum. These quantities are given by

$$\mathbf{g} = \frac{1}{c}\,\mathbf{E} \times \mathbf{B}, \tag{5}$$

$$T = \mathbf{EE} + \mathbf{BB} - \tfrac{1}{2}(E^2 + B^2)U, \tag{6}$$

$$\mathbf{F} = \rho z\mathbf{E} + \frac{1}{c}\,\mathbf{I} \times \mathbf{B}. \tag{7}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field, U the unit tensor, z the charge per unit mass (defined by $z\rho = \Sigma_k z_k \rho_k$ with z_k the charge of k per unit mass) and $\mathbf{I}$ the electrical current ($\Sigma_k \rho_k z_k \mathbf{v}_k$).

Conservation of total momentum can be expressed as

$$\frac{\partial}{\partial t}(\rho\mathbf{v} + \mathbf{g}) = -\mathrm{Div}\,(\rho\mathbf{vv} + P - T), \tag{8}$$

where P is the pressure tensor (it is actually defined by this equation). When (4) is subtracted from (8) one obtains (using also the barycentric time derivative)

$$\rho\frac{d\mathbf{v}}{dt} = -\mathrm{Div}\,P + \mathbf{F}, \tag{9}$$

the equation of motion, with the Lorentz force density $\mathbf{F}$ as a source of momentum. Clearly one can also write for (7)

$$\mathbf{F} = \Sigma_k \rho_k \mathbf{F}_k, \qquad \mathbf{F}_k = z_k(\mathbf{E} + \frac{1}{c}\,\mathbf{v}_k \times \mathbf{B}), \tag{10}$$

where $\mathbf{F}_k$ represents a force per unit mass on component k.

(*c*) *Conservation of energy*. Let us first establish a balance equation for the kinetic energy, which follows from multiplication of (9) by $\mathbf{v}$:

$$\frac{\partial \frac{1}{2}\rho v^2}{\partial t} = -\mathrm{div}\,\mathbf{J}_{\mathrm{kin}} + \sigma_{\mathrm{kin}}, \tag{11}$$

where

$$\mathbf{J}_{\mathrm{kin}} = \tfrac{1}{2}\rho v^2\mathbf{v} + P \cdot \mathbf{v}, \tag{12}$$

$$\sigma_{\mathrm{kin}} = P : \mathrm{Grad}\,\mathbf{v} + \rho z\mathbf{v} \cdot \mathbf{E} - \mathbf{i} \cdot \left(\frac{1}{c}\,\mathbf{v} \times \mathbf{B}\right) \tag{13}$$

and with

$$\mathbf{i} = \Sigma_k \rho_k z_k(\mathbf{v}_k - \mathbf{v}) = \mathbf{I} - \rho z\mathbf{v}, \tag{14}$$

which is the conduction current.

Another equation can be deduced from Maxwell's equations:

$$\frac{\partial e_f}{\partial t} = -\mathrm{div}\,\mathbf{S} - \sigma_f, \tag{15}$$

the balance equation of electromagnetic energy, with the density of electromagnetic energy e_f, the flow of electromagnetic energy (Poynting vector) $\mathbf{S}$ and a source term σ_f. These quantities are given by

$$e_f = \tfrac{1}{2}E^2 + \tfrac{1}{2}B^2, \tag{16}$$

$$\mathbf{S} = c\mathbf{E} \times \mathbf{B}, \tag{17}$$

$$\sigma_f = \mathbf{I} \cdot \mathbf{E}. \tag{18}$$

Conservation of total energy can be expressed by

$$\frac{\partial \rho e}{\partial t} = -\mathrm{div}\,\mathbf{J}_e, \tag{19}$$

where e is the total energy per unit mass and $\mathbf{J}_e$ the total energy flux. Let us now define the internal energy u and the heat flow $\mathbf{J}_q$ by splitting the two quantities which occur in (19) in the following way

$$\rho e = \tfrac{1}{2}\rho v^2 + e_f + \rho u, \tag{20}$$

$$\mathbf{J}_e = \mathbf{J}_{\mathrm{kin}} + \mathbf{S} + \rho u\mathbf{v} + \mathbf{J}_q. \tag{21}$$

If one now subtracts the balances (11) for $\frac{1}{2}\rho v^2$ and (15) for e_f from (19) one obtains the balance equation of internal energy

$$\rho\frac{du}{dt} = -\mathrm{div}\,\mathbf{J}_q - P : \mathrm{Grad}\,\mathbf{v} + \mathbf{i} \cdot \left(\mathbf{E} + \frac{1}{c}\,\mathbf{v} \times \mathbf{B}\right). \tag{22}$$

Splitting the pressure tensor into an equilibrium part pU and a viscous pressure tensor Π, and denoting the electric field measured by an observer with a velocity $\mathbf{v}$ as $\mathbf{E}'(= \mathbf{E} + (\mathbf{v}/c) \times \mathbf{B})$ one can write instead of (22)

$$\rho\frac{du}{dt} = -\mathrm{div}\,\mathbf{J}_q - \rho p\frac{d\rho^{-1}}{dt} - \Pi : \mathrm{Grad}\,\mathbf{v} + \mathbf{i} \cdot \mathbf{E}', \tag{23}$$

a form of the first law of thermodynamics.

(*d*) *The entropy law* (Gibbs' relation) has the form

$$T\frac{ds}{dt} = \frac{du}{dt} + p\frac{d\rho^{-1}}{dt} - \sum_{k=1}^{n} \mu_k \frac{dc_k}{dt}, \tag{24}$$

where s is the entropy per unit mass and μ_k the chemical potential of component k. This law is valid along the barycentric motion if deviations from equilibrium are not exceedingly large (PRIGOGINE, 1947; DE GROOT, 1951; DE GROOT and MAZUR, to be published).

(*e*) *The entropy balance equation* is obtained from (24) with the help of (2) and (23):

$$\frac{\partial \rho s}{\partial t} = -\mathrm{div}\,\mathbf{J}_s + \sigma, \tag{25}$$

where $\mathbf{J}_s$ is the entropy flux and σ the entropy production, are given by

$$\mathbf{J}_s = \rho s\mathbf{v} + (\mathbf{J}_q - \sum_{k=1}^{n} \mu_k \mathbf{J}_k)/T, \tag{26}$$

$$T\sigma = -\frac{1}{T}\mathbf{J}_q \cdot \operatorname{grad} T - \sum_{k=1}^{n} \mathbf{J}_k \cdot T \operatorname{grad} \frac{\mu_k}{T} + \mathbf{i} \cdot \mathbf{E}' - \Pi : \operatorname{Grad} \mathbf{v} \geq 0. \tag{27}$$

The latter quantity is positive definite according to the second law of thermodynamics and measures the irreversibility (it vanishes at equilibrium). It consists of a sum of products of so-called fluxes and thermodynamic forces. It is seen to contain terms arising from heat conduction, diffusion, electric conduction and viscous flow respectively. In non-equilibrium thermodynamics one establishes linear relations between these fluxes and thermodynamic forces, and one studies the properties of the so-called phenomenological coefficients which occur in these linear relations. The scheme of these coefficients can be simplified as a result of possible spatial symmetries of the system ('Curie's principle') or as a result of time reversal invariance of the equations governing the motion of the individual constituent particles which leads to the 'Onsager reciprocal relations.'

In such a way one can study various phenomena such as electric conduction, thermo-electric phenomena, thermomagnetic and galvanomagnetic effects, and the like (PRIGOGINE, 1947; DE GROOT, 1951; DE GROOT and MAZUR, to be published).

3. POLARIZABLE SYSTEMS IN AN ELECTROMAGNETIC FIELD

The formalism of the preceding section must be modified if the system can be polarized by the electromagnetic field. We give here a summary of the results for this case.

(i) *Conservation of mass* is still expressed by (1) or (2).

(ii) *Conservation of momentum.* A balance equation (4) of electromagnetic momentum can be derived from Maxwell's equations, but now with the following expressions for the quantities occurring in it

$$\mathbf{g} = \frac{1}{c}\mathbf{E} \times \mathbf{H}, \tag{28}$$

$$T = \mathbf{DE} + \mathbf{BH} + \frac{\mathbf{v}}{c}(\mathbf{P} \times \mathbf{B} - \mathbf{M} \times \mathbf{E}) - (\tfrac{1}{2}E^2 + \tfrac{1}{2}B^2 - \mathbf{M} \cdot \mathbf{B})U, \tag{29}$$

$$\mathbf{F} = \rho z\mathbf{E} + \frac{1}{c}\mathbf{I} \times \mathbf{B} + (\operatorname{Grad} \mathbf{E}) \cdot \mathbf{P} + (\operatorname{Grad} \mathbf{B}) \cdot \mathbf{M} + \frac{\rho}{c}\frac{d}{dt}(\mathbf{p} \times \mathbf{B} - \mathbf{m} \times \mathbf{E}), \tag{30}$$

where $\mathbf{p} = \rho^{-1}\mathbf{P}$ and $\mathbf{m} = \rho^{-1}\mathbf{M}$ are the electric and magnetic polarization per unit mass. Furthermore conservation of total momentum is again expressed by (8), and the equation of motion is (9). The force $\mathbf{F}$ contains the Lorentz terms encountered already in Section 1, and moreover terms involving the polarizations. The latter are called together the 'ponderomotive force.' The tensor T and the vector $\mathbf{F}$ could be defined differently since only Div $T - \mathbf{F}$ and $P - T$ (cf. equations (4) and (8)) are well-defined quantities. The form adopted here fits well into the thermodynamical description and is moreover essentially the ponderomotive force proposed by Kelvin. We shall later show that the more familiar different form, proposed by Helmholtz, is not contradictory with Kelvin's expression.

(iii) *Conservation of energy.* The balance equation (11) of kinetic energy is again obtained by multiplication of (9) by $\mathbf{v}$. After some manipulation one finds

$$\mathbf{J}_{\text{kin}} = \tfrac{1}{2}\rho v^2\mathbf{v} + P \cdot \mathbf{v} - (\mathbf{P}' \cdot \mathbf{E} + \mathbf{M}' \cdot \mathbf{B})\,\mathbf{v},$$

$$\sigma_{\text{kin}} = P : \operatorname{Grad} \mathbf{v} + \rho z\mathbf{v} \cdot \mathbf{E} - \mathbf{i} \cdot \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) - \rho\,\mathbf{E}' \cdot \frac{d\mathbf{p}'}{dt} - \rho\,\mathbf{B}' \cdot \frac{d\mathbf{m}'}{dt} + \mathbf{E} \cdot \frac{\partial \mathbf{P}'}{\partial t} + \left(\mathbf{M} \times \frac{\mathbf{v}}{c}\right) \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{M}'}{\partial t} - \left(\mathbf{P} \times \frac{\mathbf{v}}{c}\right) \cdot \frac{\partial \mathbf{B}}{\partial t}, \tag{31}$$

where the primed quantities

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B} \quad , \quad \mathbf{B}' = \mathbf{B} - \frac{1}{c}\mathbf{v} \times \mathbf{E}, \tag{32}$$

$$\mathbf{P}' = \mathbf{P} + \frac{1}{c}\mathbf{M} \times \mathbf{v} \quad , \quad \mathbf{M}' = \mathbf{M} - \frac{1}{c}\mathbf{P} \times \mathbf{v} \tag{33}$$

are the fields and polarizations as measured by an observer with velocity $\mathbf{v}$.

The balance equation of electromagnetic energy (15) can again be obtained from Maxwell's equation, with the following quantities

$$e_f = \tfrac{1}{2}E^2 + \tfrac{1}{2}B^2 - \mathbf{M}' \cdot \mathbf{B} - \frac{\mathbf{v}}{c} \cdot (\mathbf{E} \times \mathbf{M}), \tag{34}$$

$$\mathbf{S} = c\mathbf{E} \times \mathbf{H}, \tag{35}$$

$$\sigma_f = \mathbf{I} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{P}'}{\partial t} + \left(\mathbf{M} \times \frac{\mathbf{v}}{c}\right) \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{M}'}{\partial t} - \left(\mathbf{P} \times \frac{\mathbf{v}}{c}\right) \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{36}$$

Conservation of total energy is given by (19). Again by using the decompositions (20) and (21) one

gets after subtraction of (11) and (15) from (19) the balance equation of internal energy

$$\rho \frac{du}{dt} = -\operatorname{div} \mathbf{J}_q - P : \operatorname{Grad} \mathbf{v} + \mathbf{i} \cdot \mathbf{E}' + \rho\left(\mathbf{E}' \cdot \frac{d\mathbf{p}'}{dt} + \mathbf{B}' \cdot \frac{d\mathbf{m}'}{dt}\right) = -\operatorname{div} \mathbf{J}_q - \rho p \frac{d\rho^{-1}}{dt} - \Pi : \operatorname{Grad} \mathbf{v} + \mathbf{i} \cdot \mathbf{E}' + \rho\left(\mathbf{E}' \cdot \frac{d\mathbf{p}'}{dt} + \mathbf{B}' \cdot \frac{d\mathbf{m}'}{dt}\right) \tag{37}$$

the form of the first law of thermodynamics for polarized systems.

(iv) *The entropy law.* In order to establish the second law for polarized systems, let us first consider a one-component polarizable system at rest and confined to a constant volume V with surface Ω. Energy conservation is then expressed by

$$dE = dQ + dW_{\text{mech}} + dW_{\text{em}}, \tag{38}$$

where E is the total energy, dQ the heat added and dW_{mech} and dW_{em} the mechanical and the electromagnetic work performed on the system in a time dt. One has

$$\frac{dW_{\text{em}}}{dt} = -\int^{\Omega} \mathbf{S} \cdot d\Omega = -\int^{V} \operatorname{div} \mathbf{S}\, dV = \int \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\right) dV. \tag{39}$$

Inserting this and also (20) with (34) for our system, which is at rest ($\mathbf{v} = 0$), into (38) yields

$$dU = dQ + dW_{\text{mech}} + \int^{V} (\mathbf{E} \cdot d\mathbf{P} + \mathbf{B} \cdot d\mathbf{M})\, dV. \tag{40}$$

For a reversible transformation one has therefore

$$dU = T\, dS + dW_{\text{mech}}^{\text{rev}} + \int^{V} (\mathbf{E} \cdot d\mathbf{P} + \mathbf{B} \cdot d\mathbf{M})\, dV, \tag{41}$$

from which we can conclude, since the density is constant,

$$du = T\, ds + dw_{\text{mech}}^{\text{rev}} + \mathbf{E} \cdot d\mathbf{p} + \mathbf{B} \cdot d\mathbf{m}. \tag{42}$$

One can convince oneself easily that the deformation parameter of the mechanical work is only the density so that (42) becomes

$$T\, ds = du + p\, d\rho^{-1} - \mathbf{E} \cdot d\mathbf{p} - \mathbf{B} \cdot d\mathbf{m}. \tag{43}$$

For an open system this can be generalized to

$$T\, ds = du + p\, d\rho^{-1} - \mathbf{E} \cdot d\mathbf{p} - \mathbf{B} \cdot d\mathbf{m} - \textstyle\sum_i \mu_i\, dc_i, \tag{44}$$

which is the generalization of Gibbs' relation for polarized systems, valid along the centre of gravity motion. It must be noticed that $\mathbf{E}$ and $\mathbf{B}$ in (44) are the equilibrium fields.

(v) *The entropy balance equation* (25) follows from (44) after inserting (2) and (37), with the entropy flux (26) and the entropy production σ given by

$$T\sigma = -\mathbf{J}_q \cdot \frac{1}{T} \operatorname{grad} T - T \sum_{k=1}^{n} \mathbf{J}_k \cdot \operatorname{grad} \frac{\mu_k}{T} + \mathbf{i} \cdot \mathbf{E}' + \rho\left(\Delta\mathbf{E}' \cdot \frac{d\mathbf{p}'}{dt} + \Delta\mathbf{B}' \cdot \frac{d\mathbf{m}'}{dt}\right) - \Pi : \operatorname{Grad} \mathbf{v}. \tag{45}$$

As compared with (27) this formula contains two additional terms describing electric and magnetic relaxation. The quantities $\Delta\mathbf{E}'$ and $\Delta\mathbf{B}'$ are the deviations of the values of the fields, which occur in (37), from their equilibrium values which occur in (44).

(vi) *On pressure and ponderomotive force.* As remarked before, pressure and force can be defined in different ways in polarizable systems (MAZUR and PRIGOGINE, 1953; MAZUR and DE GROOT, 1956). We encountered a pressure p and a ponderomotive force $\mathbf{F}$, which was essentially the Kelvin force. One can, however, define the pressure in a different way and thus obtain a corresponding different force. In particular the pressure in a polarized state of the system is sometimes defined as the pressure p_0 measured in the system at equilibrium when one reduces the fields to zero, but keeps temperature and density constant. In order to find the relation between p_0 and the pressure p introduced before, let us introduce the Helmholtz function

$$f = u - Ts. \tag{46}$$

With the help of the Gibbs' relation for polarized systems, one has then

$$df = -s\,dT - p\,dv + \mathbf{E} \cdot d\mathbf{p} + \mathbf{B} \cdot d\mathbf{m}, \tag{47}$$

where $v = \rho^{-1}$ is the specific volume. The polarizations are related to the fields by means of the linear relationships

$$\mathbf{p}\rho = \mathbf{P} = \kappa\mathbf{E}, \tag{48}$$

$$\mathbf{m}\rho = \mathbf{M} = \chi\mathbf{H} = \{\chi/(\chi + 1)\}\mathbf{B}, \tag{49}$$

where κ and χ are the electric and magnetic susceptibilities, which are scalar quantities in isotropic systems (the case which we shall study here). Integrating (47) at constant temperature and density, one obtains with (48) and (49)

$$f = f_0 + \tfrac{1}{2}(\kappa v)^{-1} p^2 + \tfrac{1}{2}(\chi v)^{-1}(\chi + 1) m^2, \tag{50}$$

where f_0 is the free energy at temperature T, density $\rho = v^{-1}$ and fields zero. Differentiating this relation with respect to the specific volume, at constant temperature and polarizations, gives

$$p = p_0 + \tfrac{1}{2}E^2\frac{\partial \kappa v}{\partial v} + \tfrac{1}{2}H^2\left(\frac{\partial \chi v}{\partial v} + \chi^2\right), \qquad (51)$$

the connexion between p and p_0 which we wished to derive.

In the equation of motion occurred the expression grad p—$\mathbf{F}$. Should we prefer to use p_0 instead of p, then we must introduce also a different force $\mathbf{F}_H$ such that

$$\text{grad}\, p - \mathbf{F} = \text{grad}\, p_0 - \mathbf{F}_H. \qquad (52)$$

This equality gives with (51) and (30) for the equilibrium case

$$\mathbf{F}_H = \rho^z\mathbf{E} + \frac{1}{c}\mathbf{I} \times \mathbf{B} - \tfrac{1}{2}E^2\,\text{grad}\,\kappa$$

$$-\tfrac{1}{2}\,\text{grad}\times\left(E^2 v\frac{\partial \kappa}{\partial v}\right) - \tfrac{1}{2}H^2\text{grad}\,\chi - \tfrac{1}{2}\,\text{grad}\left(H^2 v\frac{\partial \chi}{\partial v}\right), \qquad (53)$$

the so-called Helmholtz ponderomotive force, which is seen here to correspond to the use of p_0 as pressure.

(vii) *Relativistic theory*. The preceding considerations were all non-relativistic. It may, however, be remarked that a fully relativistic theory exists, of which the non-relativistic approximation is identical to the formalism summarized above (KLUITENBERG, DE GROOT and MAZUR, 1953; KLUITENBERG and DE GROOT, 1954 and 1955).

4. TWO-FLUID THEORY

It has been suggested that the macroscopic equations which describe the behaviour of a mixture may have to be fundamentally altered if the transfer of momentum between some components of the mixture is, for some reason or other, inhibited (PRIGOGINE and MAZUR, 1951; MAZUR and PRIGOGINE, 1951). In such a case these parts of the system are sometimes referred to as 'fluids.' This 'inhibition hypothesis' is then supposed to have as a consequence that separate Gibbs' relations must be written down for the various fluids since approximate statistical equilibrium is no longer established around the centre of gravity motion of the mixture but only around the macroscopic velocity of each fluid separately:

$$T_\alpha\frac{ds_\alpha}{dt} = \frac{du_\alpha}{dt} + p_\alpha\frac{ds_\alpha^{-1}}{dt} - \sum_i \mu_{\alpha i}\frac{dc_{\alpha i}}{dt}\,.\ (\alpha = 1, 2, \ldots) \qquad (54)$$

In this equation α numbers the fluids, which may each consist of a number of chemical components numbered by the index i. (The time derivative must be counted with respect to the average motion of fluid α.) The existence of different temperatures T_α for the various fluids α leads to the possibility of a scalar heat flow between the fluids. We shall in the following study a system which we shall for convenience suppose to be at rest, but in which we allow, in contrast with the preceding sections, a chemical reaction to take place between the various components, which do not all belong to the same fluid. This case is interesting since because both the relaxation phenomenon referred to in Section 1 (scalar flow of heat) and the chemical reaction are scalar processes they can couple even in isotropic systems, thus giving rise to interference phenomena. Let us develop the theory of such a system by writing down the conservation and entropy laws.

(*a*) *Conservation of mass*. We can write for the mass equation

$$\frac{\partial \rho_{\alpha i}}{\partial t} = \nu_{\alpha i} J_c, \qquad (55)$$

where $\rho_{\alpha i}$ is the density of component i of fluid α. At the right-hand side no divergence of a flow appears, since the system was supposed to be at rest. However, a source term $\nu_{\alpha i}J_c$ arises, which describes the production of matter of the species αi by the chemical reaction. The quantity $\nu_{\alpha i}$ divided by the molecular mass $M_{\alpha i}$ of αi is proportional to the stoichiometric coefficient of αi in the reactional equation (these quantities are counted positive if the substance αi appears there at the right-hand side, negative if it appears at the left-hand side). The factor J_c is called the chemical reaction rate. Conservation of mass can be expressed as

$$\Sigma_{\alpha,i}\nu_{\alpha i} = 0, \qquad (56)$$

because the left-hand side of (55) summed over α and i vanishes in a system at rest (constant density ρ).

(*b*) *Conservation of energy* is expressed by

$$\frac{\partial \rho e}{\partial t} = \frac{\partial \Sigma_\alpha \rho_\alpha e_\alpha}{\partial t} = 0. \qquad (57)$$

For a system with electromagnetic forces (but no polarization) or with conservative external forces (such that the potential energy is conserved in the chemical reaction), this equation leads to an internal energy balance equation for fluid α of the form

$$\frac{\partial \rho_\alpha u_\alpha}{\partial t} = -\gamma_\alpha, \qquad (58)$$

where the source terms γ_α satisfy

$$\Sigma_\alpha \gamma_\alpha = 0. \tag{59}$$

(*c*) *The entropy law* (54) becomes for a system at rest

$$\frac{\partial \rho_\alpha s_\alpha}{\partial t} = \frac{1}{T_\alpha}\frac{\partial \rho_\alpha u_\alpha}{\partial t} - \Sigma_i \frac{\mu_{\alpha i}}{T_\alpha}\frac{\partial \rho_{\alpha i}}{\partial t}. \tag{60}$$

(*d*) *The entropy balance equation* is obtained by inserting (55) and (58) into (60). This yields

$$\frac{\partial \rho_\alpha s_\alpha}{\partial t} = \sigma_\alpha, \tag{61}$$

where σ_α is the entropy production in fluid α, given by

$$\sigma_\alpha = -\frac{\gamma_\alpha}{T_\alpha} - \frac{1}{T_\alpha}\Sigma_i \mu_{\alpha i}(T_\alpha) v_{\alpha i} J_c, \tag{62}$$

where we have explicitly indicated that $\mu_{\alpha i}$ is a function of the temperature T_α.

The total entropy production for a two-fluid system ($\alpha = 1, 2$) is then, using also (59), and the notation J_u for $\gamma_1 = -\gamma_2$,

$$\sigma = -J_u\left(\frac{1}{T_1} - \frac{1}{T_2}\right) - J_c\left(\frac{A_1(T_1)}{T_1} + \frac{A_2(T_2)}{T_2}\right), \tag{63}$$

where

$$A_\alpha(T_\alpha) = \Sigma_i \mu_{\alpha i}(T_\alpha) v_{\alpha i}. \qquad (\alpha = 1,2) \tag{64}$$

We can write (63) as a sum of products of fluxes J_u and J_c and thermodynamic forces X_u and X_c

$$\sigma = J_u X_u + J_c X_c, \tag{65}$$

where

$$X_u = \frac{1}{T_2} - \frac{1}{T_1} \equiv \Delta\left(\frac{1}{T}\right) \tag{66}$$

and

$$\begin{aligned} X_c &= -\frac{A_1(T_1)}{T_1} - \frac{A_2(T_2)}{T_2} \\ &= -\frac{A_1(T_1) + A_2(T_1)}{T_1} - \Delta\left(\frac{A_2}{T}\right) \\ &= -\frac{A(T_1)}{T_1} - \Delta\left(\frac{A_2}{T}\right), \end{aligned} \tag{67}$$

with the symbol Δq standing for the difference of a quantity q with index 2 and the same quantity with index 1. The quantity A is the so-called chemical affinity

$$A = \Sigma_{\alpha,i} \mu_{\alpha i} v_{\alpha i} \tag{68}$$

of the reaction.

The linear phenomenological relations for the fluxes and forces of (65) are

$$J_u = L_{uu} X_u + L_{uc} X_c, \tag{69}$$

$$J_c = L_{cu} X_u + L_{cc} X_c, \tag{70}$$

with the Onsager relation

$$L_{uc} = L_{cu}. \tag{71}$$

The phenomenological coefficient L_{uu} describes the scalar heat conduction, L_{cc} the chemical reaction velocity and L_{uc} and L_{cu} characterize mutual interference phenomena between the relaxation process and the chemical reaction.

If the chemical reaction takes place at a much quicker rate than the energy flow, a kind of stationary state would be reached in which the reaction rate J_c would vanish. Then from (70) we have

$$\left(\frac{X_c}{X_u}\right)_{J_c=0} = -\frac{L_{cu}}{L_{cc}}. \tag{72}$$

Let us compare this quantity, which gives the thermodynamic chemical force X_c per unit temperature force X_u in the state of vanishing chemical reaction rate, with another quantity which characterizes another important physical situation, namely the state of equal temperature ($X_u = 0$). Then one has from (69) and (70)

$$\left(\frac{J_u}{J_c}\right)_{X_u=0} = \frac{L_{uc}}{L_{cc}}. \tag{73}$$

This quantity, which we shall abbreviate as u^*, is therefore the energy transfer per unit reaction rate at the isothermal state. Both quantities (72) and (73) could in principle be measured and it could be checked whether the Onsager relation (71) which with (72) and (73) can be expressed as

$$\left(\frac{X_c}{X_u}\right)_{J_c=0} = -u^* \tag{74}$$

is experimentally verified.

PRIGOGINE and GÉHÉNIAU (1946) have studied the example of a gas plasma, where the chemical reaction is the ionization of a molecule A into an ion I and an electron e:

$$A \rightleftarrows I + e. \tag{75}$$

The fluid $\alpha = 1$ consists of the molecules A and the ions I with temperature T_1 and molecular masses M and $M - m$, the fluid 2 consists of the electrons with temperature T_2 and mass m. The stoichiometric coefficients of reaction (75) for molecules, ions and electrons are clearly -1, $+1$ and $+1$ respectively, and the quantities $\nu_{\alpha i}$ can therefore be chosen as -1, $1 - M/m$ and M/m respectively. The plasma is supposed to be in such a state that the degree of ionization is approximately the same which would exist at equilibrium, i.e. at a temperature between T_1 and T_2. Furthermore, since the heat content of the

first fluid consisting of the heavy particles is much greater that the heat content of the second fluid, we may consider T_1 as practically the temperature of chemical equilibrium. This means that the chemical affinity A is zero for the temperature T_1 and (67) becomes thus

$$X_e = -\Delta\left(\frac{A_2}{T}\right) = -\Delta\left(\frac{v_e\mu_e}{T}\right) = -\frac{M}{m}\Delta\left(\frac{\mu_e}{T}\right), \tag{76}$$

because A_2 contains only quantities referring to the second fluid, which consists of the electrons e. The last expression can be written as

$$X_e = \frac{M}{m}\left(h_e\frac{\Delta T}{T^2} - v_e\frac{\Delta p_e}{T}\right), \tag{77}$$

with h_e the partial specific enthalpy, v_e the partial specific volume and p_e the partial pressure of the electrons. The Onsager relation (74) becomes with (66) and (77)

$$\frac{\Delta p_e}{\Delta T} = \frac{h_e m - u^* M}{v_e m T}, \tag{78}$$

which, with the perfect gas law $p_e v = kT/m$, becomes

$$\frac{\Delta p_e}{p_e} = \frac{h_e m - u^* M}{kT}\frac{\Delta T}{T}, \tag{79}$$

giving the deviation of the electron pressure from its equilibrium (Guldberg and Waage) value. For the verification of (79) the quantities $\Delta p_e/\Delta T$ at zero reaction rate $J_e = 0$ and the 'energy of transfer' u^* at the isothermal state should be measured. One can express the deviation Δp_e alternatively as a deviation of the reaction rate constant

$$K = \frac{N_i N_e}{N_a} = \frac{N_e^2}{N_a}, \tag{80}$$

(with N_i, N_e and N_a the number densities of the ions, electrons and molecules) from its equilibrium value

$$K_0 = \frac{N_{io}N_{eo}}{N_a} = \frac{N_{eo}^2}{N_a}. \tag{81}$$

Indeed from the perfect gas law,

$$p_e = N_e k T_2 \tag{82}$$

and the same for equilibrium

$$p_{eo} = N_e k T_1 \tag{83}$$

it follows with (79), (80) and (81) that

$$\frac{K - K_0}{K_0} = 2\left(\frac{\Delta p_e}{p_e} - \frac{\Delta T}{T}\right) = 2\left(\frac{h_e m - u^* M - kT}{kT}\right)\left(\frac{\Delta T}{T}\right), \tag{84}$$

which gives the relative deviation of the chemical rate constant from its equilibrium value.

If in the preceding derivation polarizable media would have been studied, one would have found in the entropy source strength terms related to electric and magnetic relaxation besides the relaxation term and the chemical term appearing in (63).

Finally it may be noted that in two-fluid systems, which are not at rest, PRIGOGINE and MAZUR (1951) have found that a characteristic 'friction term' arises in the entropy source strength, which depends on the relative velocity of the two fluids, and which is absent in the case of ordinary mixtures. On the other hand in two-fluid mixtures terms, arising from diffusion of chemical components belonging to different fluids with respect to each other, are missing from the entropy production. Such terms arise only in ordinary mixtures, where the transfer of momentum between all chemical components is uninhibited.

REFERENCES

GROOT S. R. DE (1951) *Thermodynamics of Irreversible Processes* North Holland Publishing Company; Amsterdam, Interscience Publishers, New York.

GROOT S. R. DE and MAZUR P., *Non-equilibrium Thermodynamics*, North Holland Publishing Company, Amsterdam. To be published.

KLUITENBERG G. A. and GROOT S. R. DE (1954) *Physica, Amsterdam* **20**, 199; (1955) **21**, 148, 169.

KLUITENBERG G. A., GROOT S. R. DE and MAZUR P. (1953) *Physica, Amsterdam* **19**, 689, 1079;

MAZUR P. and GROOT S. R. DE (1956) *Physica, Amsterdam* **22**, 657.

MAZUR P. and PRIGOGINE I. (1951) *Physica, Amsterdam* **17**, 680.

MAZUR P. and PRIGOGINE I. (1953) *Mém. Acad. Roy. Belg., Cl.Sc.* **28**, 1.

PRIGOGINE I. (1947) *Thermodynamique des Phénomènes Irréversibles*, Desoer, Liège.

PRIGOGINE I. and GÉHÉNIAU J. (1946) *C.R. Acad. Sci. (Paris)*, **222**, 1085.

PRIGOGINE I. and MAZUR P. (1951) *Physica, Amsterdam* **17**, 661.

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 195 to 205. Pergamon Press Ltd. Printed in Northern Ireland

FOKKER-PLANCK EQUATION FOR A PLASMA WITH A CONSTANT MAGNETIC FIELD*

N. ROSTOKER and M. N. ROSENBLUTH†

Abstract—Starting from the Liouville equation, a chain of equations is obtained by integrating out the co-ordinates of all but one, two-etc. particles. One particle is singled out. All other particles are considered to be initially in thermal equilibrium. For the time evolution of the distribution function of the 'singled out' particle an equation is obtained whose asymptotic form is of the usual Fokker-Planck type. It is characterized by a frictional drag force that slows the particle down and a fluctuation tensor that speeds it up and produces diffusion in velocity space. The objective of the calculation is to determine these quantities for a plasma consisting of electrons and protons in a constant external magnetic field.

The chain of equations contains two dimensionless parameters $\lambda = \omega_c/\omega_p$ and $g = 1/n L_D^3$. A solution for the s-body correlation function is obtained in the form $f_s = f_s^{(0)}(\lambda) + g f_s^{(1)}(\lambda) +$ etc. $f_s^{(0)}$ and $f_s^{(1)}$ have been determined to all orders of λ.

The frictional drag consists of a part due to collisions and a part due to plasma wave emission. When $\lambda \ll 1$ the modification of the collisional part due to the magnetic field is negligible. There is a significant change in the properties of plasma waves of wavelength greater than the Larmor radius which modifies the force due to plasma wave emission. When $\lambda \gg 1$ the force due to plasma wave emission disappears. The collisional force is altered to the extent that the maximum impact parameter is sometimes the Larmor radius instead of the Debye length, or something in between. A more interesting modification obtains for the particular case of a slow ion moving perpendicular to the field. It is due to repeated collisions with fast electrons. This collisional force is of a qualitatively different form, but the quantitative modification is not large. In addition to the drag force anti-parallel to the velocity of the particle, there is a collisional force anti-parallel to the Lorentz force. This force arises because the particle and its 'shield cloud' are spiralling about field lines. The force on the particle is equal and opposite to the centripetal force acting on the 'shield cloud.' It is much smaller than the Lorentz force.

The main result is that to the lowest order in g, the frictional drag and fluctuation tensor are slowly varying functions of λ.

1. INTRODUCTION

THE usual kinetic theory of gases does not apply to a plasma because of the long range character of Coulomb forces. In the treatment of ROSENBLUTH, MACDONALD and JUDD (1957) the infrequent large angle scatterings are treated as collisions. The effect of the many small angle deflexions is accounted for by means of a macroscopic field.

GASIOROWICZ, NEUMAN and RIDDELL (1956) have considered a test-particle problem. The frictional drag was determined from the response of the plasma to the test-particle. Collisions between plasma particles that produce large deflexions were neglected. Fluctuation effects were obtained by a Holtzmark-type calculation.

Plasma wave effects were not obtained with either of these methods. Making use of the random-phase approximation BOHM and PINES (1952) obtained a frictional drag for fast particles that is due to plasma wave emission. It is of the same order of magnitude as the collisional effects.

In this paper an ensemble of plasmas will be considered. The density in phase space satisfies the Liouville equation. By integrating out the co-ordinates of all particles but one, but two, etc. a chain of equations is obtained for the one-body, two-body etc. functions. KADOMTSEV (1958) and CHAN-MOU TCHEN (to be published) have previously discussed plasma kinetics in terms of this chain. They consider the general problem of transport theory. In this paper only the test-particle problem will be considered. The loss in generality is compensated for by a substantial gain in tractability.

2. REDUCTION OF THE LIOUVILLE EQUATION

Consider a gas of electrons and infinite mass randomly distributed ions. It is a simple matter to generalize the results for this model to include finite mass ions. Let $X_i = (\mathbf{X}_i, \mathbf{V}_i)$ be the position and velocity co-ordinates of the i-th electron. N electrons and N ions are contained in a volume V. For an

* This work was done under a contract between General Atomic and the Texas Atomic Energy Research Foundation.

† John Jay Hopkins Laboratory for Pure and Applied Science, General Atomic Division of General Dynamics Corporation, San Diego, California.

ensemble of similar plasmas the density in phase space $D(X_1, X_2, \ldots, X_N; t)$ satisfies the Liouville equation,

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{N}\left[\mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} - \frac{e}{m}\left(\mathbf{E}(\mathbf{x}_i, t) + \frac{1}{c}\mathbf{v}_i \times \mathbf{B}\right) \cdot \frac{\partial}{\partial \mathbf{v}_i}\right]\right\} D = 0. \quad (1)$$

Coulomb forces only are considered so that

$$\mathbf{E}(\mathbf{x}_i, t) = e \sum_{j=1}^{N}{}' \frac{\partial}{\partial \mathbf{x}_i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} \quad (2)$$

where the prime indicates that the term $i = j$ is to be omitted. $\mathbf{B}$ is a constant externally applied magnetic field. The Liouville operator is symmetric with respect to the interchange of the co-ordinates of any two electrons. Therefore if D is initially symmetric it will remain so. In the test particle problem, one electron is singled out initially so that D will not be symmetric with respect to interchange of its co-ordinates with any other electron. If ions are included D will not be symmetric with respect to the interchange of the co-ordinates of an electron and an ion.

The s-body function is defined as

$$f_s(X_1, X_2, \ldots, X_s; t) = V^s \int D\, dX_{s+1} \ldots dX_N. \quad (3)$$

By taking moments of equation (1) a chain of equations is obtained. Assuming D has complete symmetry it is

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{s}\left(\mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} - \frac{e}{mc}\mathbf{v}_i \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{v}_i}\right) - \frac{e^2}{m} \times \sum_{ij=1}^{s}{}' \frac{\partial}{\partial \mathbf{x}_i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} \cdot \frac{\partial}{\partial \mathbf{v}_i}\right\} f_s - \frac{ne^2}{m} \times \sum_{i=1}^{s} \int \left(\frac{\partial}{\partial \mathbf{x}_i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_{s+1}|}\right) \cdot \frac{\partial f_{s+1}}{\partial v_i}\, dX_{s+1} = 0. \quad (4)$$

where $n = N/V$ and both N and V are large. In the limit that $|\mathbf{x}_i - \mathbf{x}_j| \to \infty$ the third term becomes negligible. Omitting this term it can be verified by direct substitution that a solution of equation (4) is

$$f_s^{(0)} = \prod_{i=1}^{s} f^{(0)}(X_i, t), \quad (5)$$

where

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} - \frac{e}{m}(\mathbf{E}_M{}^{(0)} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial}{\partial \mathbf{v}}\right] f^{(0)} = 0, \quad (6)$$

and $$\mathbf{E}_M{}^{(0)} = ne \int \left(\frac{\partial}{\partial \mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) f^{(0)}(X', t)\, dX'$$

is the macroscopic field. If, instead of taking this limit we take the limit $e \to 0$, $m \to 0$, $n \to \infty$ such that e/m and ne remain constant, the result is the same. In this limit the plasma becomes a continuous fluid. Equation (6) which is the collisionless Boltzmann equation is then a precise description. If the effects due to particle individuality are small a suitable calculation procedure is to make an expansion in terms of some parameter proportional to e where m and $1/n$ are considered to be of the same order. The natural units of the problem are $1/\omega_p$ for time and L_D for length where $\omega_p = (4\pi ne^2/m)^{1/2}$ is the plasma frequency and $L_D = (\Theta/4\pi ne^2)^{1/2}$ is the Debye length. $3\Theta/2$ is the mean energy per electron. If Equation (4) is re-written in these units, the significant dimensionless parameters of the problem are self-evident.

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{s}\left(\mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} - \lambda \mathbf{v}_i \times \mathbf{e}_z \cdot \frac{\partial}{\partial \mathbf{v}_i}\right) - \frac{g}{4\pi} \times \sum_{ij=1}^{s}{}' \frac{\partial}{\partial \mathbf{x}_i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} \cdot \frac{\partial}{\partial \mathbf{x}_i}\right\} f_s - \frac{1}{4\pi} \times \sum_{i=1}^{s} \int \left(\frac{\partial}{\partial \mathbf{X}_i} \frac{1}{|\mathbf{x}_i - \mathbf{x}_{s+1}|}\right) \cdot \frac{\partial f_{s+1}}{\partial \mathbf{v}_i}\, dX_{s+1} = 0 \quad (4.1)$$

$\mathbf{e}_z$ is a unit vector in the direction of $\mathbf{B}$. $\lambda = \omega_c/\omega_p$ where $\omega_c = eB/mc$ is the cyclotron frequency and $g = 1/nL_D{}^3 = 0[e]$. The calculation procedure is to expand in powers of g and retain λ to all orders. The expansion in powers of g is similar to the MAYER (MAYER and MAYER, 1940) cluster expansion of equilibrium statistical mechanics.

$$f_s = \prod_{i=1}^{s} f(X_i, t) + \sum_{p} [\pi f(X_i t)] P(X_j, X_k; t) + \sum_{pp} [\pi f(X_i t)]\, P(X_j, X_k; t) P(X_l, X_m; t) + \sum_{j} [\pi f(X_i t)] T(X_j, X_k, X_l; t) + \ldots \text{etc.} \quad (7)$$

The second term is summed over pairs, the third over pairs of pairs and the fourth over triplets. In the present treatment the series will be terminated after the second term, or up to second order in g. In the lowest order the solution of Equation (5) is such that the s-body function can be expressed in terms of one body functions. In the next order, the s-body function can be expressed in terms of one and two-body functions. We substitute

$$f_s = \prod_{i=1}^{s} f(X_i t) + \sum_{p} [\pi f^{(0)}(X_i, t)] P(X_i X_k; t) \quad (8)$$

into equation (4) and retain only terms of order less

than g^2. P is of order g and $f = f^{(0)} + f^{(1)}$ where $f^{(1)}$ is of order g. The result is

$$\sum_{i=1}^{s} [\pi f(X, t)] \left\{ \left[\frac{\partial}{\partial t} + \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} - \frac{e}{m} \times \left(\mathbf{E}_M(\mathbf{x}_i, t) + \frac{1}{c} \mathbf{v}_i \times \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{v}_i} \right] f(X_i, t) - \frac{ne}{m} \times \int \mathbf{E}(\mathbf{x}_i, \mathbf{x}) \cdot \frac{\partial}{\partial \mathbf{v}_i} P(X_i, X;\ t)\, dX \right\}$$

$$\frac{1}{2} \sum_{i,j=1}^{s}{}' [\pi f^{(0)}(X, t)] \left\{ \frac{\partial}{\partial t} P(X_i X_j;\ t) + \left[\mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} P - \frac{e}{m} (\mathbf{E}_M{}^{(0)}(\mathbf{x}_i, t) + \frac{1}{c} \mathbf{v}_i \times \mathbf{B}) \cdot \frac{\partial}{\partial \mathbf{v}_i} P - \frac{ne}{m} \frac{\partial f^{(0)}}{\partial \mathbf{v}_i} \cdot \int \mathbf{E}(\mathbf{x}_i, x) P(X_j, X;\ t)\, dX - \frac{e}{m} f^{(0)}(X_j, t) \mathbf{E}(\mathbf{x}_i, \mathbf{x}_j) \cdot \frac{\partial}{\partial \mathbf{v}_j} f^{(0)}(X_i t) \right] + \begin{bmatrix} i \to j \\ j \to i \end{bmatrix} \right\} = 0, \quad (9)$$

where $\mathbf{E}(\mathbf{x}, \mathbf{x}') = e \frac{\partial}{\partial \mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|}$

and $\mathbf{E}_M(\mathbf{x}, t) = n \int \mathbf{E}(\mathbf{x}, \mathbf{x}') f(X', t)\, dX'.$

The cases $s = 1$ and $s = 2$ give equations that define $f(X, t)$ and $P(X, X';\ t)$. Because of the interchangeability of particle co-ordinates it is clear that equation (8) satisfies equation (4) for $s > 2$. As the only restriction on $f^{(0)}(X, t)$ is that it satisfy equation (6) the equation for $P(X, X';\ t)$ is not very tractable. Some progress has been made in obtaining physically meaningful solutions for $\mathbf{B} = 0$. (KADOMTSEV, 1958; CHAN-MOU TCHEN, to be published).

If one particle is singled out, the reduction of the Liouville equation can be accomplished in a similar way. There will be two kinds of s-body distribution functions according to whether or not the test particle is included. The two coupled chains of equations have been solved up to second order in g. To this order, the solutions for both kinds of s-body functions may be expressed in terms of one- and two-body functions only; i.e. closure is achieved as in the case where D has full symmetry. The results will be stated in terms of the equations that define the one- and two-body functions. In addition ions of finite mass have been included.

The zero-order one body functions are solutions of the equation

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{q_i}{m_i} \left(\mathbf{E}_M{}^{(0)} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{v}} \right\} f^{(0)}(X, t) = 0 \quad (10)$$

where

$$\mathbf{E}_M{}^{(0)} = n \sum_j \int \mathbf{E}_j(\mathbf{x}, \mathbf{x}') f_j{}^{(0)}(X, t)\, dX'$$

$\mathbf{E}_j(\mathbf{x}, \mathbf{x}')$ is the Coulomb electric field at $\mathbf{x}$ from a particle of species j at $\mathbf{x}'$. $j = 1$ means electrons and $j = 2$ means ions. q_i is the charge of a particle and m_i is its mass.

$f_j{}^{(0)}(X, t)$ means the zero-order one-body distribution for a field particle. We shall assume the field particles are initially in thermal equilibrium so that

$$f_j{}^{(0)}(X, t) = \frac{1}{(2\pi v_j{}^2)^{3/2}} \exp(-v^2/2v_j{}^2), \quad (11)$$

$$\mathbf{E}_M{}^{(0)} = 0,$$

and $m_j v_j{}^2 = \Theta$ defines the thermal velocities. $w_i{}^{(0)}(X, t)$ means the zero-order one-body distribution for the test particle. It also satisfies equation (10). The general solution is

$$w_i{}^{(0)}(X, t) = \int \Omega(X_0)\, \delta[X - X_0(t)]\, dX_0 \quad (12)$$

X_0 means the initial position and velocity co-ordinates. $\Omega(X_0)$ is an arbitrary function normalized such that $\frac{1}{V} \int \Omega(X_0)\, dX_0 = 1$. $X_0(t)$ are the time-dependent orbits in a magnetic field. Assuming the magnetic field in the z-direction

$$\mathbf{x}_0(t) = \mathbf{e}_x [x_0 + (v_{x0}/\omega_i) \sin \omega_i t + (v_{y0}/\omega_i) \times (1 - \cos \omega_i t)] + \mathbf{e}_y [y_0 + (v_{y0}/\omega_i) \times \sin \omega_i t - (v_{x0}/\omega_i)(1 - \cos \omega_i t)] + \mathbf{e}_z [z_0 + v_{z0} t]$$

and $\mathbf{v}_0(t) = d\mathbf{x}_0(t)/dt$.

There are two kinds of two-body correlation functions. For the field particles $\mathbf{P}_{ij}(X, X';\ t)$ satisfies the equation

$$\frac{D}{Dt} \bar{P}_{ij}(X, X';\ t) + \frac{nq_i}{m_i} \frac{\partial f_i{}^{(0)}}{\partial \mathbf{v}} \cdot \sum_l \int \mathbf{E}_l(\mathbf{x}, \mathbf{x}'') \bar{P}_{jl}(X', X'')\, dX'' + \frac{nq_j}{m_j} \frac{\partial f_j{}^{(0)}}{\partial \mathbf{x}'} \cdot \sum_l \int E_l(\mathbf{x}, ' \mathbf{x}'') \bar{P}_{il}(X, X'')\, dX'' + \frac{q_i}{m_i} f_j{}^{(0)}(v') \mathbf{E}_j(\mathbf{x}, \mathbf{x}') \cdot \frac{\partial}{\partial \mathbf{v}} f_i{}^{(0)} + \frac{q_j}{m_j} f_i{}^{(0)}(v) \mathbf{E}_i(\mathbf{x}', \mathbf{x}) \cdot \frac{\partial}{\partial \mathbf{v}'} f_j{}^{(0)} = 0 \quad (13)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \mathbf{v}' \cdot \frac{\partial}{\partial \mathbf{x}'} + \omega_i \mathbf{v} \times \mathbf{e}_z \cdot \frac{\partial}{\partial \mathbf{v}} + \omega_j \mathbf{v}' \times \mathbf{e}_z \cdot \frac{\partial}{\partial \mathbf{v}'}$$

and $\omega_i = q_i B/m_i c$ is the cyclotron frequency.

If the field particles are initially in thermal equilibrium, the solution of equation (13) is

$$\bar{P}_{ij}(X, X';\ t) = -\frac{q_i}{\Theta} f_i^{(0)}(v) f_j^{(0)}(v')\, \bar{\Phi}_j(\mathbf{x}, \mathbf{x}') \quad (14)$$

where

$$\bar{\Phi}_j(\mathbf{x}, \mathbf{x}') = \frac{q_j}{|\mathbf{x} - \mathbf{x}'|} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|}{L_D}\right)$$

$$\frac{1}{L_D^2} = \sum_j \frac{1}{L_j^2}$$

$$\frac{1}{L_j^2} = \frac{4\pi n q_j^2}{m_j v_j^2}.$$

If the particle of species i is the test particle the equation for $P_{ij}(X, X'; t)$ is

$$\frac{D}{Dt} P_{ij} + \frac{n q_i}{m_j} \frac{\partial f_i^{(0)}}{\partial \mathbf{v}} \cdot \sum_l \int E_l(\mathbf{x}', \mathbf{x}'') P_{il}(X, X'';\ t)\, dX'' = \frac{q_i}{m_i} f_j^{(0)}(v') \frac{\partial \bar{\Phi}_j(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{x}} \cdot \frac{\partial w_i^{(0)}}{\partial \mathbf{v}} \quad (15)$$

To solve this equation it is convenient to introduce the Green's function $G_{ij}(X, X';\ t)$ defined by

$$P_{ij}(X, X';\ t) = \int \Omega(X_0) \mathbf{G}_{ij}(X, X';\ t) \cdot \frac{\partial}{\partial \mathbf{v}} \delta[X - X_0(t)]\, dX_0 \quad (16)$$

Equation (15) can thus be transformed to a set of coupled equations for $\mathbf{G}_{ij}$,

$$\frac{D}{Dt} \mathbf{G}_{ij} + \omega_i (\mathbf{e}_z \times \mathbf{G}_{ij}) = \frac{q_i}{m_i} f_j^{(0)}(v') \frac{\partial \bar{\Phi}_j(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{x}} + \frac{q_j}{m_j} \left(\frac{\partial f_j^{(0)}}{\partial \mathbf{v}'}\right) \cdot \frac{\partial}{\partial \mathbf{x}'} \mathbf{A}_i(X, \mathbf{x}';\ t),$$

and

$$\nabla'^2 \mathbf{A}_i(X, \mathbf{x}';\ t) = -4\pi n \sum_l q_l \int \mathbf{G}_{il}(X, X';\ t)\, d\mathbf{v}' \quad (17)$$

The one body function up to order g^2 for field particles may be expressed as

$$f_j(X', t) = f_j^{(0)}(v') + \frac{1}{V} \times \int w_i^{(0)}(X, t)\, \delta f_{ij}(X, X';\ t)\, dX \quad (18)$$

$\delta f_{ij}(X, X';\ t)$ is also of the nature of a Green's function. It means the change in the distribution at X', t due to a test particle at X. It satisfies the set of coupled equations

$$\frac{D\, \delta f_{ij}}{Dt} = \frac{q_j}{m_j} \frac{\partial f_j^{(0)}}{\partial \mathbf{v}'} \cdot \frac{\partial \Phi_i(X, \mathbf{x}';\ t)}{\partial \mathbf{x}'}, \quad (19)$$

and

$$\nabla'^2 \Phi_i(X, \mathbf{x}';\ t) = -4\pi q_i\, \delta(\mathbf{x}' - \mathbf{x}) - 4\pi n \times \sum_l q_l \int \delta f_{il}(X, X';\ t)\, d\mathbf{v}'.$$

Equations (17) and (19) are similar in type. They can be solved in a straightforward fashion by taking Fourier and Laplace transforms. It is only necessary to specify initial conditions. We shall assume that $\mathbf{G}_{ij}(X, X';\ 0) = \delta f_{ij}(X, X';\ 0) = 0$ which means that initially the test particle is not correlated with any other particle and does not possess a shield cloud.

The equation for the test-particle one-body function to order g^2 is of the Fokker-Planck type.

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{q_i}{m_i c} \mathbf{v} \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{v}}\right) w_i + \frac{1}{m_i} \frac{\partial}{\partial \mathbf{v}} \left(\mathbf{F}_i' w_i^{(0)}\right) + \frac{1}{m_i} \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}} \left(\mathsf{T}_i w_i^{(0)}\right) = 0 \quad (20)$$

The frictional drag force $\mathbf{F}_i'$ and the fluctuation tensor T_i are completely determined by the Green's functions.

$$\mathbf{F}_i' = \mathbf{F}_i - \frac{\partial}{\partial \mathbf{v}} \cdot \mathsf{T}_i \quad (21)$$

$$\mathbf{F}_i = n q_i \sum_j \int \mathbf{E}_j(\mathbf{x}, \mathbf{x}')\, \delta f_{ij}(X, X';\ t)\, dX' \quad (22)$$

$$\mathsf{T}_i = n q_i \sum \int \mathbf{E}_j(\mathbf{x}, \mathbf{x}') \mathbf{G}_{ij}(X, X';\ t)\, dX' \quad (23)$$

The starting point of this calculation is the Liouville equation. It possesses the property of time-reversal invariance. The expansion procedure preserves this property. Equation (20) differs from the usual Fokker-Planck equation because $\mathbf{F}_i'$ and T_i are functions of time and the equation has the property of time reversal invariance. If instead of substituting the correct values of $\mathbf{F}_i$ and T_i, the asymptotic values as $t \to \infty$ are substituted, equation (20) will be a conventional Fokker-Planck equation. The time-reversal invariance will then be destroyed.

VOL. 2 1961

3. TEST-PARTICLE CALCULATIONS FOR $\mathbf{B} = 0$

The Laplace-Fourier transform of $\delta f_{ij}(X, X'; t)$ can be determined by solving equations (19) with no approximations, i.e.

$$\delta f_{ij}(\mathbf{k}, p) = \frac{(-1)^{i+j+1}}{2np} \frac{i(\mathbf{k}\cdot\mathbf{v}')f_j^{(0)}(v')}{[p + i\mathbf{k}\cdot(\mathbf{v}' - \mathbf{v})][(kL_D)^2 + W]}; \quad (24)$$

p and $\mathbf{k}$ are the Laplace and Fourier transform variables and

$$W = -\tfrac{1}{2}\sum_j \int_0^\infty dt \exp\left\{-[p - i(\mathbf{k}\cdot\mathbf{v})]t\right\} \times \left\{\frac{d}{dt}\exp\left[-\frac{(kv_jt)^2}{2}\right]\right\}.$$

The denominator in equation (24) has poles at $p = 0$, at $p = i\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')$ and at the roots of $(kL_D)^2 + W = 0$. The $p = 0$ pole yields the asymptotic solution. The pole $p = i\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')$ produces a term that describes the removal of depleted charge from the vicinity of the test particle to infinity. Field particles move away from the test charge with their unperturbed velocities relative to the test charge. Ultimately the field particle distribution achieves a permanent charge depletion around the test charge. The equation $(kL_D)^2 + W = 0$ has an infinite set of roots that have been discussed by LANDAU (1946) (in a different form). All but one root produces terms that damp out in a negligible time. The last of the 'Mohicans' is for $p = i(\mathbf{k}\cdot\mathbf{v}) \pm i\omega(k) - \epsilon_L(k)$, where

$$\omega(k) \cong \omega_p[1 + 3(kL_D)^2 \ldots],$$

$$\epsilon_L(k) = \frac{\sqrt{\pi}}{8}\frac{\omega_p}{(kL_D)^3}\exp\left(-\frac{1}{4(kL_D)^2}\right). \quad (25)$$

This pole gives rise to a term that damps rapidly for $kL_D > 1$ and very slowly for $kL_D < 1$. If the Fourier inversion is also carried out, a term is obtained that decays like $1/\sqrt{\omega_p t}$. The cause of this decay is the dispersion of $\omega(k)$.

The present treatment will be limited to the asymptotic form of the Fokker-Planck equation, and therefore to the $p = 0$ pole.

The frictional drag from the fluid response of the plasma is obtained from equations (22) and (24). It has only a component parallel to the velocity of the test particle

$$F_\parallel = \frac{4\pi i q^2}{(2\pi)^2}\int_0^\infty k\,dk \int_{-1}^{1}\frac{\mu\,d\mu W(\mu v/v_1)}{(kL_D)^2 + W(\mu v/v_1)}. \quad (26)$$

The function $W(x) = -\operatorname{Re}(x) + i\operatorname{Im}(x)$ is plotted in Fig. 1. The integral is divergent. That the present procedure breaks down at short distances is apparent from the case $\Omega(X_0) = \delta(X_0)$, i.e. a zero-velocity test charge, in which case

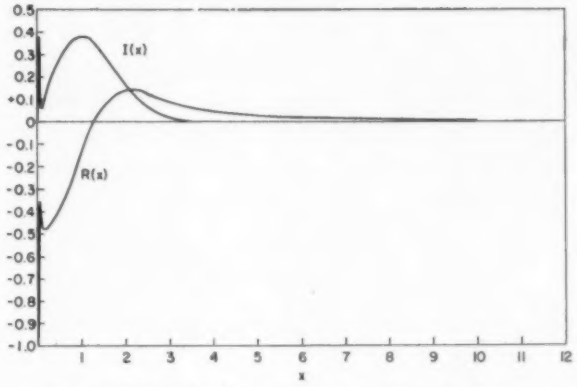


FIG. 1.—The W-function $W(x) = -\operatorname{Re}(x) + i\operatorname{Im}(x)$.

$$t \xrightarrow{\lim} \infty \quad f_j(X, t) = f_j^{(0)}\left[1 - \frac{q_iq_j}{m_jv_j^2}\frac{\exp(-|\mathbf{x}|/L_D)}{|\mathbf{x}|}\right].$$

If $|x| < q^2/\Theta$, f_j will not be positive in the case $q_i = q_j$. For $q_i = -q_j$, a quantum-mechanical treatment is required to avoid the divergence. We might expect the present treatment to be appropriate for collective effects. For distances less than the interparticle spacing $(3/4\pi n)^{1/3}$, there can be no collective effects and the binary collision method should be more appropriate. (ROSENBLUTH, MACDONALD and JUDD, 1957). The result is the same if we simply employ the cut-off $k_{\max} \cong \Theta/q^2$ in the present treatment.

Since $|\operatorname{Re}(x)| \leq 1$ and $|\operatorname{Im}(x)| < 1$ the denominator in equation (26) can be expanded in the domain $kL_D > 1$.

This leads to a contribution of the collision type,

$$F_\parallel = \frac{\Theta F_0}{2}\ln D\sum_j \frac{1}{m_j}\frac{\partial}{\partial v}\left[\frac{1}{v}\left(\operatorname{erf}\frac{v}{\sqrt{2}v_j}\right)\right]$$

$$\cong -\frac{2}{3\sqrt{\pi}}F_0\ln D\sum_j (v/v_j) \quad \text{for} \quad v \ll v_j$$

$$\sim -\frac{F_0}{2}\ln D\sum_j (v_j/v)^2 \quad \text{for} \quad v \gg v_j \quad (27)$$

$F_0 = q^2/L_D^2$, and $D = k_{\max}\ L_D = 4\pi g^{-1}$. For the domain $kL_D < 1$, it is convenient to write equation (26) in the form

$$F_\parallel = -\frac{1}{v}\int S(k, \boldsymbol{\Omega})\,dk\,d\boldsymbol{\Omega}$$

where

$$S(k, \boldsymbol{\Omega}) = v\frac{(kL_D)^2}{2\pi^2} \times \frac{k^3\mu\operatorname{Im}(\mu v/v_1)}{[(kL_D)^2 - \operatorname{Re}(\mu v/v_1)]^2 + [\operatorname{Im}(\mu v/v_1)]^2},$$

and $d\mathbf{\Omega} = 2\pi\, d\mu$. The integrand is of a resonant character for $v \gg v_1$. In this case $\mathrm{Re}\,(x) \cong 1/2x^2 > 0$ and $|\mathrm{Im}\,(x)| \ll 1$. The resonance corresponds to the emission of plasma waves. For a given k, the emission takes place at the angle $\cos\theta = \mu = v_1/\sqrt{2}v(kL_D)$. After carrying out the $\mathbf{\Omega}$-integration, the result is

$$\bar{S}(k, v) = \frac{1}{4\pi}\int S(k, \mathbf{\Omega})\, d\mathbf{\Omega} = q^2\omega_p{}^2/4\pi kv,$$

for $v/v_1 > 1/\sqrt{2}(kL_D)$. The drag force due to plasma wave emission is

$$F_{\parallel} = -\frac{4\pi}{v}\int_{k=v_1/\sqrt{2}(vL_D)}^{k=1/L_D} \bar{S}(k, v)\, dk \cong -\frac{F_0}{2}\left(\frac{v_1}{v}\right)^2 \ln\frac{v}{v_1}. \quad (28)$$

The rate of emission of energy from the test particle is $F_{\parallel}v$, or $4\pi\bar{S}(k, v)\, dk$ from wave numbers between k and $k + dk$. The total emission per unit volume and wave number for a plasma with a Maxwell distribution of particles is

$$Q(k) = 4\pi n\sum_j \int_{v>v_1/\sqrt{2}(kL_D} \bar{S}(k, v) f_j^{(0)}(v)\, d^3v = A(k)\Theta k^2/2\pi^2, \quad (29)$$

where $A(k) = 2\epsilon_L(k)$ is the energy absorption constant and $\epsilon_L(k)$ is the LANDAU damping given by equation (25). Equation (29) is simply a statement of Kirchhoff's law of radiation with the classical Rayleigh-Jeans distribution $\Theta k^2/2\pi^2$.

The tensor T_i has only two independent components $T_{\parallel}$ and $T_{\perp}$, which may be computed by the same techniques. The collision contributions from the domain $kL_D > 1$ are

$$T_{\parallel} = \frac{v_i^2}{v} F_{\parallel} \quad (30)$$

$$T_{\perp} = -\frac{F_0}{4}\frac{v_i^2}{v}\ln D\sum_j \left(1 + \frac{\partial}{\partial v}\frac{v_j^2}{v}\right)\mathrm{erf}\left(\frac{v}{\sqrt{2}v_j}\right). \quad (31)$$

These terms are due to field-particle individuality. The total drag force is

$$F'_{\parallel} = F_{\parallel} - \frac{\partial T_{\parallel}}{\partial v} - 2\frac{(T_{\parallel} - T_{\perp})}{v}. \quad (32)$$

The collision contribution is

$$F_{\parallel}{}' = \frac{\Theta F_0}{2}\ln D\sum_j \left(\frac{1}{m_j} + \frac{1}{m_i}\right)\frac{\partial}{\partial v}\left[\frac{1}{v}\mathrm{erf}\left(\frac{v}{\sqrt{2}v_j}\right)\right]. \quad (33)$$

By comparing equation (33) with equation (27), it is apparent that the consequence of the individuality of the field particles is to replace the field-particle mass $1/m_j$ by the effective mass $1/m_j + 1/m_i$; i.e. recoil of the test particle is introduced.

The plasma-wave contributions from the domain $kL_D <$ are

$$T_{\parallel} = \frac{v_i^2}{v} F_{\parallel}$$

$$T_{\perp} = -T_{\parallel}/2 - (F_0 v_i{}^2/4v)\left[1 - \frac{1}{2}(v_1/v)^2 \ldots\right]. \quad (34)$$

The plasma-wave part of the total drag force is, for $v \gg v_1$,

$$F_{\parallel}{}' = -\frac{F_0}{2}\left(\frac{v_1}{v}\right)^2 \ln\frac{v}{v_1} - \frac{F_0}{2}\left(\frac{v_i}{v}\right)^2 + F_0\frac{v_i{}^2 v_1{}^2}{v^4}. \quad (35)$$

The latter two terms are small corrections due to field-particle individuality. The first term may be interpreted as being the result of spontaneous emission of plasma waves, the second as stimulated emission and the third as absorption.

4. EFFECT OF A CONSTANT EXTERNAL MAGNETIC FIELD

The procedure for solving equations (17) and (19) is similar to the case where $\mathbf{B} = 0$. It is however more complicated because the unperturbed orbits are spirals about the magnetic field lines instead of straight lines. The details of this calculation will be omitted here (ROSENBLUTH and ROSTOKER, 1959). In this section the calculation of $\mathbf{F}_i$ and T_i will be considered making use of the asymptotic results for $\delta f_{ij}(X, X')$ and $\mathbf{G}_{ij}(X, X')$. $\mathbf{F}_i$ and T_i are functions of $\mathbf{v}$ the test particle velocity. It is convenient to employ cylindrical coordinates $(v_{\perp}, \beta, v_z)$ for $\mathbf{v}$, since the test particle spirals about the z-direction. The angular variable β is defined so that $v_x = -v_{\perp}\sin\beta$, $v_y = v_{\perp}\cos\beta$. All vectors and dyadics will be expressed in terms of the unit vectors $(\mathbf{e}_{\rho}, \mathbf{e}_{\beta}, \mathbf{e}_z)$ where

$$\mathbf{e}_x = \mathbf{e}_{\rho}\cos\beta - \mathbf{e}_{\beta}\sin\beta$$
$$\mathbf{e}_y = \mathbf{e}_{\rho}\sin\beta + \mathbf{e}_{\beta}\cos\beta$$

In terms of these unit vectors

$$\mathbf{v} = v_{\perp}\mathbf{e}_{\beta} + v_z\mathbf{e}_z \quad (36)$$
$$\mathbf{F}_i = F_{\rho}\mathbf{e}_{\rho} + F_{\beta}\mathbf{e}_{\beta} + F_z\mathbf{e}_z$$
$$\mathsf{T}_i = T_{\rho\rho}\mathbf{e}_{\rho}\mathbf{e}_{\rho} + T_{\beta\beta}\mathbf{e}_{\beta}\mathbf{e}_{\beta} + T_{zz}\mathbf{e}_z\mathbf{e}_z + T_{\rho\beta}(\mathbf{e}_{\rho}\mathbf{e}_{\beta} - \mathbf{e}_{\beta}\mathbf{e}_{\rho}) + T_{\beta z}(\mathbf{e}_{\beta}\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_{\beta}) + T_{z\rho}(\mathbf{e}_z\mathbf{e}_{\rho} - \mathbf{e}_{\rho}\mathbf{e}_z).$$

The components $F_\rho, F_\beta \ldots T_{z\rho}$ are functions of $v_\perp$ and v_z, but not of β. They are as follows:

$$\mathbf{F}_i = 2\left(\frac{q_i L_D}{2\pi}\right)^2 \int d^3\mathbf{k} \sum_{m=-\infty}^{\infty} \frac{\boldsymbol{\gamma}_i(\mathbf{k}, m)}{[(kL_D)^2 + W(\mathbf{k}, m)]} \quad (37)$$

$$\mathsf{T}_i = v_i^2\left(\frac{q_i L_D}{2\pi}\right)^2 \sum_j \int \frac{d^3\mathbf{k}}{[(kL_D)^2 + 1]} \times \sum_{m=-\infty}^{\infty} \frac{N_j(\mathbf{k}, m)\boldsymbol{\Gamma}_i(\mathbf{k}, m)}{[(kL_D)^2 + W(\mathbf{k}, m)]} \quad (38)$$

$$N_j(\mathbf{k}, m) = i\int_0^\infty dt \exp\,[i(k_z v_z - m\omega_i)t] \exp\,[-\xi_j(t)] \quad (39)$$

$$W(\mathbf{k}, m) = -\frac{1}{2}\int_0^\infty dt \exp\,[i(k_z v_z - m\omega_i)t]\frac{d}{dt} \times \exp\,[-\xi_j(t)] = 1 + \frac{1}{2}(k_z v_z - m\omega_i)\sum_j N_j(\mathbf{k}, m)$$

and

$$\xi_j(t) = \frac{1}{2}(k_z v_j t)^2 + (k_\perp \bar{a}_j)^2(1 - \cos\omega_j t) \quad (40)$$

Cylindrical co-ordinates $(k_\perp, \alpha, k_z)$ are employed for $\mathbf{k}$; $\bar{a}_j = v_j/\omega_j$ is the mean Larmor radius for field particles of species j.

The components of the vector $\boldsymbol{\gamma}_i$ and the tensor $\boldsymbol{\Gamma}_i$ contain Bessel functions of argument $r = k_\perp a_i$ where $a_i = v_\perp/\omega_i$ is the Larmor radius of the test-particle. These components are

$$\begin{aligned}
\gamma_\rho &= k_\perp J_m(r)J_m'(r)\\
\gamma_\beta &= i(m/a_i)[J_m(r)]^2\\
\gamma_z &= -ik_z[J_m(r)]^2\\
\Gamma_{\rho\rho} &= ik_\perp^2[J_m'(r)]^2\\
\Gamma_{\beta\beta} &= i[(m/a_i)J_m(r)]^2\\
\Gamma_{zz} &= i[k_z J_m(r)]^2\\
\Gamma_{\rho\beta} &= k_\perp(m/a_i)J_m(r)J_m'(r)\\
\Gamma_{\beta z} &= -ik_z(m/a_i)[J_m(r)]^2\\
\Gamma_{z\rho} &= k_\perp k_z J_m(r)J_m'(r).
\end{aligned} \quad (41)$$

J_m is the Bessel function of the first kind of order m and $J_m'(r) = dJ_m(r)/dr$. Equations (37) to (41) were obtained by solving equations (17) and (19) exactly.

Of the nine components given by equation (37) and (38) only six are independent. The asymptotic solutions as $t \to \infty$, $\delta f_{ij}(X, X')$ and $\mathbf{G}_{ij}(X, X')$ must satisfy a detailed balance condition. This condition becomes apparent if we consider a Maxwell distribution for the test particle, i.e. $\Omega(X_0) = f_i^{(0)}(v)$ and $w_i^{(0)}(X, t) = f_i^{(0)}(v) = w_i(X, t)$. In this case the two-body function

$$w_i(X)\left[f_j^{(0)}(X') + \frac{1}{V}\int w_i^{(0)}(X)\,\delta f_{ij}(X, X')\,dX\right] + \int \Omega(X_0)\mathbf{G}_{ij}(X, X')\cdot\frac{\partial}{\partial\mathbf{v}}\delta[X - X_0(t)]\,dX_0,$$

must be the thermal equilibrium two-body function $f_i^{(0)}(v)f_j^{(0)}(v') + \bar{P}_{ij}(X, X')$. The detailed balance condition is therefore

$$f_i^{(0)}(v)\left[\delta f_{ij}(X, X') - \frac{\mathbf{v}}{v_i^2}\cdot\mathbf{G}_{ij}(X, X')\right] = \bar{P}_{ij}(X, X'). \quad (42)$$

Equation (42) implies that

$$\mathbf{F}_i - (\mathsf{T}_i\cdot\mathbf{v}/v_i^2) = 0, \quad (43)$$

which reduces the number of independent components of $\mathbf{F}_i$ and T_i from nine to six. The tensor T_i contains a symmetric and an anti-symmetric part. Only the symmetric part need be retained in the term $\frac{1}{m_i}\frac{\partial}{\partial\mathbf{v}}\cdot\frac{\partial}{\partial\mathbf{v}}\cdot(\mathsf{T}_i w_i^0)$ of equation (20). The total frictional drag force is given by equation (21). The components of $\mathbf{F}_i'$ are

$$\begin{aligned}
F_\rho' &= F_\rho + \frac{\partial T_{\rho\beta}}{\partial v_\perp} - \frac{\partial T_{z\rho}}{\partial v_z}\\
F_\beta' &= F_\beta - \frac{\partial T_{\beta\beta}}{\partial v_\perp} - \frac{\partial T_{\beta z}}{\partial v_z} - \frac{(T_{\beta\beta} - T_{\rho\rho})}{v_\perp}\\
F_z' &= F_z - \frac{\partial T_{\beta z}}{\partial v_\perp} - \frac{\partial T_{zz}}{\partial v_z} - \frac{T_{\beta z}}{v_\perp}.
\end{aligned} \quad (44)$$

The components $T_{\rho\beta}$ and $T_{z\rho}$ only play a part in the determination of the radial force F_ρ'. The radial force is anti-parallel to the Lorentz force. It is smaller than the Lorentz force by a factor g so that if the present expansion procedure converges rapidly, the radial force and the anti-symmetric part of T_i may be neglected.

To make any further progress in the calculation of $\mathbf{F}_i$ and T_i it is necessary to carry out the integrations of equations (37) and (38) approximately. The method consists of dividing up the $\mathbf{k}$-space into regions where asymptotic forms of $N_j(\mathbf{k}, m)$ and $W(\mathbf{k}, m)$ are applicable. If the results are insensitive to how the division is made they are considered to be acceptable. The asymptotic forms of $N_j(\mathbf{k}, m)$ are as follows.

(i) If $|k_z| \gg 1/\bar{a}_j$ the term $\exp -\frac{1}{2}(k_z v_j t)^2$ annihilates the integrand in a time $t \ll 1/\omega_j$. We may therefore expand $\cos\omega_j t = 1 - (\omega_j t)^2/2$ and retain only the first two terms. The asymptotic form is

$$N_j(\mathbf{k}, m) \cong \frac{1}{kv_j}\,U\left(\frac{m\omega_i - k_z v_z}{kv_j}\right), \quad (45)$$

and

$$W(\mathbf{k}, m) \cong W\left(\frac{k_z v_z - m\omega_i}{k v_j}\right),$$

where

$$U(x) = i \int_0^\infty (\exp -t^2/2)(\exp -ixt)\, dt$$

and

$$W(x) = 1 + \frac{x}{2} U(-x) + \frac{x}{2}\sqrt{\frac{m_2}{m_1}}\, U\left(-x\sqrt{\frac{m_2}{m_1}}\right)$$
$$= -\mathrm{Re}\,(x) + i\,\mathrm{Im}\,(x).$$

(ii) If $|k_z \ll 1/\bar{a}_j$ and $k_\perp \ll 1/\bar{a}_j$, $\exp[-(k_\perp \bar{a}_j)^2(1 - \cos \omega_j t)] \cong 1$ and

$$N_j(\mathbf{k}, m) \cong \frac{1}{|k_z|\, v_j}\, U\left(\frac{m\omega_i - k_z v_z}{|k_z|\, v_j}\right) \qquad (46)$$

(iii) If $|k_z| \ll 1/\bar{a}_j$ and $k_\perp \gg 1/\bar{a}_j$, $\exp -\xi_j(t) \ll 1$ except in the neighbourhood of $\omega_j t = 2n\pi$. In the neighbourhood of these points we may set $t \doteq \frac{2n\pi}{\omega_j} + \tau$ and $\cos \omega_j t \cong 1 - (\omega_j \tau)^2/2$. The result is

$$N_j(\mathbf{k}, m) \cong \frac{1}{k v_j}\left\{U\left(\frac{m\omega_i - k_z v_z}{k v_j}\right) + 2i\left[U_1\left(\frac{m\omega_i - k_z v_z}{k v_j}\right)\right] \times \sum_{n=1}^{\infty} \exp\left[-\frac{(k_z \bar{a}_j 2\pi n)^2}{2}\right] \times \exp\left[2\pi i n\left(\frac{k_z v_z - m\omega_i}{\omega_j}\right)\right]\right\} \qquad (47)$$

where $U_1(x) = \sqrt{\pi/2}\, \exp(-x^2/2$. The second term is of a resonant character. When $k_z v_z - m\omega_i = s\omega_j$ where $s = 0, \pm 1, \pm 2$, etc. the sum will be

$$\sum_{n=1}^{\infty} \exp\left[-\frac{(k_z \bar{a}_j 2\pi n)^2}{2}\right] \cong \frac{1}{2\sqrt{2\pi}|k_z|a_j} \gg 1 \qquad (48)$$

If $v_z \gg v_j$ the resonances will be sharp and the spacing narrow. The width of a resonance will be $\delta(k_z \bar{a}_j) \sim (k_z \bar{a}_j)(v_j/v_z)$ and the spacing $\Delta(k_z \bar{a}_j) \sim (v_j/v_z)$. The fraction of k_z-space occupied by resonances will be of order $k_z \bar{a}_j \ll 1$. The resonances may be neglected in this case. If $v_z \ll v_j$ the resonances will be broad and the spacing large. Assuming $\omega_1 \neq \omega_2 \times$ an integer, the only case in which the resonant term is significant is when $i = j$ and $v_z \ll v_j$. In this case the sum is given by equation (48).

The subdivision of $\mathbf{k}$-space is illustrated in Fig. 2. $|W(\mathbf{k}, m)| \lesssim 1$ except in the resonance regions indicated by hatching. If $|W(\mathbf{k}, m)| < 1$ and $kL_D > 1$,

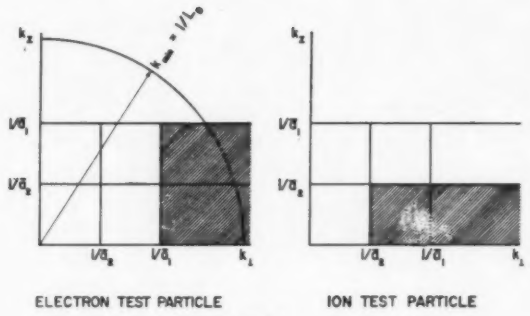


FIG. 2.—Subdivision of $\mathbf{k}$-space.

the denominator in equations (37) and (38) may be expanded

$$\frac{1}{(kL_D)^2 + W(\mathbf{k}, m)} \cong \frac{1}{(kL_D)^2}\left\{1 - \frac{W(\mathbf{k}, m)}{(kL_D)^2}\right\} \qquad (49)$$

The contributions from all domains where this expansion is permissible are designated collision-type, and resonance-type otherwise.

Consider a typical integral such as

$$F_z = -4\pi\left(\frac{q_i L_D}{2\pi}\right)^2 \int k_\perp\, dk_\perp \int dk_z i k_z \times \sum_{m=-\infty} \frac{J_m^2(k_\perp a_i)}{[(kL_D)^2 + W(\mathbf{k}, m)]} \qquad (50)$$

For the collision contribution, the domain of integration is bounded on the outside by a sphere of radius $k_{\max} = \sqrt{k_\perp^2 + k_z^2}$ and on the inside by a sphere of radius L_D if $L_D < \bar{a}_1$. The resonance region indicated in Fig. 2 must be omitted. This however involves a small angular region so that it may be included in the domain of integration provided the asymptotic form of equation (45) is employed. After expanding the denominator in equation (50), the sum can be expressed as

$$\sum_{m=-\infty}^{\infty} \frac{J_m^2(k_\perp a_i)}{[(kL_D)^2 + W(\mathbf{k}, m)]} \cong \frac{1}{(kL_D)^2}\left\{1 + \frac{1}{2(kL_D)^2} \sum_{j,m} \int_0^\infty dt\, J_m^2(k_\perp a_i) \times [\exp i(k_z v_z - m\omega_i)t]\left[\frac{d}{dt} \exp -\frac{(kv_j t)^2}{2}\right]\right\}$$

The first term can be omitted because it is even in k_z. In the second term

$$\sum_m J_m^2(k_\perp a_i) \exp -im\omega_i t = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \times \exp\{ik_\perp a_i[\cos(\alpha - \omega_i t) - \cos\alpha]\} \cong \frac{1}{2\pi}\int_0^{2\pi} d\alpha \exp(ik_\perp v_\perp t \sin\alpha),$$

making use of the fact that $\exp -\frac{(kv_j t)^2}{2}$ annihilates the integrand before $\omega_i t$ departs appreciably from zero. Thus equation (50) is reduced to

$$F_z \cong (q_i/2\pi L_D)^2 \int d^3\mathbf{k} \frac{ik_z}{k^4} \times \sum_j (\mathbf{k}\cdot\mathbf{v}/kv_j) U(-\mathbf{k}\cdot\mathbf{v}/kv_j)$$
$$= -4\pi(v_z/v)(q_i/2\pi L_D)^2 \int_{k_{\min}}^{k_{\max}} \frac{dk}{k} \int_0^1 d\mu\, \mu^2 \times \sum_j (v/v_j) U_1(-\mu v/v_j)$$
$$= \frac{F_0}{2}\frac{v_z}{v} \ln D \sum_j \frac{\partial}{\partial v}\left(\frac{v_j^2}{v} \operatorname{erf} \frac{v}{\sqrt{2}v_j}\right). \qquad (50.1)$$

If $L_D < \bar{a}_1$, $D = k_{\max} L_D$. If $L_D \not< \bar{a}_1$ the calculation is the same except that the domain of integration is defined by the limits of applicability of the asymptotic form given by equation (45). Thus $D = k_{\max} b_{\max}$ where $b_{\max} = 1/k_{\min}$ is given in Table 1. Similarly the

TABLE 1.—MAXIMUM IMPACT PARAMETER FOR SIMPLE COLLISION TERMS

Electron test particle	Ion test particle	Region of applicability
L_D	L_D	$L_D < \bar{a}_1$
$\bar{a}_1$	L_D	$\bar{a}_1 < L_D < \bar{a}_2$
$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_2 < L_D$

collision contributions to all other components may be determined. The results can be expressed most conveniently in a co-ordinate system for which T_i is diagonal, neglecting the anti-symmetric part. The unit vectors $\mathbf{e}_\beta$ and $\mathbf{e}_z$ are replaced by $\mathbf{e}_v$ and $\mathbf{e}_\sigma$ where

$$\mathbf{e}_v = \mathbf{v}/v = (v_\perp/v)\mathbf{e}_\beta + (v_z/v)\mathbf{e}_z$$
$$\mathbf{e}_\sigma = (\mathbf{v} \times \mathbf{B} \times \mathbf{v})/vv_\perp B$$
$$= -(v_z/v)\mathbf{e}_\beta + (v_\perp/v)\mathbf{e}_z. \qquad (51)$$

In terms of $\mathbf{e}_\rho$, $\mathbf{e}_v$, $\mathbf{e}_\sigma$,

$$\mathbf{F}_i \cong F\mathbf{e}_v \qquad (52)$$
$$\mathsf{T}_i \cong T_{\parallel}\mathbf{e}_v\mathbf{e}_v + T_\perp(\mathbf{e}_\rho\mathbf{e}_\rho + \mathbf{e}_\sigma\mathbf{e}_\sigma).$$

The equations for $F_{\parallel}$, $T_{\parallel}$ and $T_\perp$ are the same as in the zero magnetic field case except for the definition of D. The radial force has been neglected. The result for F_ρ is

$$F_\rho \cong -(F_0 v_\perp \omega_i/\pi v^2 k_{\min})\{1 - \tfrac{1}{2}\sum_j [(v/v_j) + (v_j/v)] U_R(v/v_j)\}$$
$$\cong (F_0 v_\perp \omega_i/\pi v^2 k_{\min}) \sum_j (v_j/v)^2$$
for $v \gg v_j$
$$\cong (F_0 v_\perp \omega_i/3\pi v^2 k_{\min}) \sum_j (v/v_j)^2$$
for $v \ll v_j$ (53)

$\mathbf{e}_\rho$ has been defined so that it is always directed from the guiding centre of the particle towards the particle. Therefore $\mathbf{F}_\rho$ is always anti-parallel to the Lorentz force. The magnitude of the Lorentz force is $|F_L| = m_i v_\perp \omega_i$ so that

$$|F_\rho|/|F_L| \cong (2/k_{\max} k_{\min} L_D^2)(v_i/v)^2 \sum_j (v_j/v)^2$$
for $v \gg v_j$
$$\cong (2/k_{\max} k_{\min} L_D^2) \sum_j (v_i/v_j)^2$$
for $v \ll v_j$.

Since $(1/k_{\max} k_{\min} L_D^2) \leq 1/k_{\max} L_d = g/4\pi$, and $g \ll 1$ is the basis of the entire calculation, F_ρ may be neglected. Similarly it can be demonstrated that the components $T_{\rho\beta}$ and $T_{z\rho}$ are negligible.

The radial force has a simple physical interpretation. The polarized cloud around the test charge gives the test charge an effective mass. It can be estimated by considering the energy of the test charge $\frac{1}{2}m_i v^2 + q_i(\Phi - \Phi_0)$ where $\Phi - \Phi_0$ is the charge in potential produced by the test-charge at the location of the test-charge. If the collision contribution is determined in the same sense that F_ρ was calculated,

$$\Phi - \Phi_0 \cong \frac{4\pi q_i}{(2\pi)^3} L_D^2 \times \int d^3\mathbf{k}\left[\frac{1}{(kL_D)^2 + W} - \frac{1}{(kL_D)^2}\right] \qquad (54)$$

For a slow test charge $\Phi - \Phi_0$ is proportional to v^2 so that the energy is $\frac{1}{2}\bar{m}_i v^2$ where

$$\bar{m}_i = m_i + \frac{1}{3\pi}\frac{q_i^2}{k_{\min} L_D^2} \sum_j (1/v_j^2). \qquad (55)$$

The additional mass $\delta m_i = (q_i^2/3\pi k_{\min} L_D^2) \sum_j (1/v_j^2)$ may be regarded as an inertial mass due to the polarized cloud. If it follows the motion of the test charge, there must be a centripetal force acting on it of magnitude.

$$F_\rho = \delta m_i v_\perp^2/a_i = (F_0 v_\perp \omega_i/3\pi v^2 k_{\min}) \sum_j (v/v_j)^2.$$

This force must be produced by a radial asymmetry of polarization and therefore an equal force must act outward on the test charge. The result agrees with equation (53).

There are additional long wavelength collision contributions when $L_D > \bar{a}_1$ and resonance contributions. For the remaining contributions we shall consider only the limiting cases

$$v_\perp = 0; \quad v_z/v_j \ll 1 \quad v_z/v_j \gg 1$$
$$v_z = 0; \quad v_\perp/v_j \ll 1 \quad v_\perp/v_j \gg 1,$$

and calculate only $F_{\parallel}$. These calculations are of the nature of a survey where the objective is simply to see if there are any large effects due to the magnetic field.

The collision terms can be added together and expressed in the form

$$F_{\parallel}^{(i)} = -(2/3\sqrt{\pi})F_0 \sum_j (\ln k_{\max} b_{ij})(v/v_j) \quad \text{for} \quad v \ll v_j$$
$$= -(F_0/2) \sum_j (\ln k_{\max} b_{ij})(v_j/v)^2 \quad \text{for} \quad v \gg v_j. \tag{56}$$

When $L_D \ll \bar{a}_j$, $b_{ij} = L_D$. When $L_D \gg \bar{a}_j$ and $v_\perp = 0$ $b_{ij} \cong (L_D \bar{a}_j)^{1/2}$ if $v_z \ll v_j$ and $b_{ij} = \bar{a}_j$ if $v_z \gg v_j$. When $L_D \gg \bar{a}_j$ and $v_z = 0$

$$b_{ij} \cong \begin{bmatrix} (\bar{a}_1 L_D)^{1/2}, & a_1 \\ \bar{a}_2, & (a_2 L_D)^{1/2} \end{bmatrix} \quad \text{if} \quad v_\perp \ll v_j,$$

and

$$b_{ij} \cong \begin{bmatrix} (\bar{a}_1 L_D)^{1/2}, & (\bar{a}_1 \bar{a}_2 L_D)^{1/3} \\ (\bar{a}_2^2 L_D)^{1/3}, & (\bar{a}_2 L_D)^{1/2} \end{bmatrix} \quad \text{if} \quad v_\perp \gg v_j.$$

In general the maximum impact parameter is something between the Debye length and a Larmor radius. The shield cloud of the test particle has a complex form so that it is not easy to see what the particular value of b_{ij} should be in an intuitive way. Equation (56) accounts for all collision terms except in the case of a slow ion moving perpendicular to the field. In this case the frictional drag is of a qualitatively different form:

$$-F_{\parallel}/F_0 = (1/2\sqrt{2\pi})\left(\ln \sqrt{\frac{m_2}{m_1}}\right)(\ln L_D/\bar{a}_2)(v_\perp/v_1) \tag{57}$$

for $v_\perp < v_2$ and $(1/2\sqrt{2\pi})(\ln v_1/v_\perp)(\ln L_D/\bar{a}_2)(v_\perp/v_1)$ for $v_2 \ll v_\perp \ll v_1$. These terms come from the region of **k**-space where $|k_z| \ll 1/\bar{a}_2$, $k_\perp \ll 1/\bar{a}_2$.

The forces due to plasma-wave emission are significantly modified by the magnetic field. If $L_D \gg \bar{a}_2$ they disappear entirely. If $L_D \ll \bar{a}_1$ and $v_\perp = 0$, the modification is that $-F_{\parallel}/F_0 = (1/2)(\ln \bar{a}_1/L_D)(v_1/v_z)^2$ instead of $(1/2)(\ln v_z/v_1)(v_1/v_z)^2$. The reason is that in the presence of a magnetic field, the plasma-wave Cerenkov effect takes place only for wavelengths $1/\bar{a}_1 < k < 1/L_D$ instead of $0 < k < 1/L_D$. When there is no magnetic field the leading term of the dispersion relation for plasma waves is $\omega(k) = \omega_p$. The phase velocity is ω_p/k. Plasma waves are emitted by the test charge mainly at the angle $\cos\theta = \omega_p/kv = (v_1/\sqrt{2}\, v_z k L_D)$. This leads to the usual Cerenkov shock front. In the presence of a magnetic field the leading term of the dispersion relation is $\omega(k) = \omega_p$ for $1/\bar{a}_1 < k < 1/L_D$, and $\omega(k) = \omega_p |k_z|/k = \omega_p \cos\theta$ (BERNSTEIN, 1958) for $0 < k < 1/\bar{a}_1$. The resonance now takes place for $\cos\theta = \omega_p \cos\theta/kv$ or $\omega_p = kv$, independent of angle. This plasma wave resonance of the second kind produces a negligible drag on a particle moving in the direction of the field. The effect is to replace the lower limit of integration in equation (28) by $k = 1/\bar{a}_1$.

For a particle moving perpendicular to the field the plasma wave drag is

$$F_{\parallel} = -(2F_0/3\pi) \ln (L_D v_\perp/\bar{a}_1 v_1)(v_1/v_\perp)^2. \tag{58}$$

It is entirely due to plasma-wave resonance of the second kind.

The results obtained with a magnetic field depend only logarithmically on the way in which k-space was divided up. $F_{\parallel}(\lambda)$ is a slowly varying function of $\lambda = L_D/\bar{a}$. If either of these conditions were violated, the calculation procedure would be of doubtful validity.

In a thermonuclear machine where the magnetic field is produced by currents within the plasma $(L_D/\bar{a}_1)^2 = B^2/4\pi n m_1 c^2$ where $B^2/4\pi \cong n\Theta$. Therefore $(L_D/\bar{a}_1)^2 \cong \Theta/m_1 c^2 < 1$ for $\Theta < 510$ keV. In a mirror machine where B is produced by external coils it is possible to have $L_D > \bar{a}_2$ for sufficiently low density n. This case may also arise in astrophysical problems. The less extreme case $\bar{a}_1 < L_D < \bar{a}_2$ is more likely. We have not listed results for this case, but they can easily be obtained by applying the same methods.

If the cluster expansion is continued, i.e. $f_s = f_s^{(0)} + g f_s^{(1)} + g^2 f_s^{(2)} \ldots$, it is probable that $f_s^{(2)}$ will contain the factor $\ln g$. This speculation was made some time ago by BOGOLUBOV (1946) for a thermal equilibrium cluster expansion and appears in the recent results of ABE (1959). The expansion in terms of g is not a power series. However for the kind of plasma that is of current interest for thermonuclear machines, g is very small. For example if $\Theta = 100$ eV, $n = 10^{16}$ cm^{-3}, $g \sim 10^{-5}$ so that a calculation to order g^2 should give satisfactory quantitative results.

Acknowledgements—The authors are indebted to DR. CHAN-MOU TCHEN, DR. W. DRUMMOND and PROF. D. PINES for constructive discussions.

REFERENCES

ABE R. (1959) *Progr. Theor. Phys. (Japan)* **21**, 475.
BERNSTEIN I. (1958) *Phys. Rev.* **109** 10.

Bogolubov N. (1946) *J. Phys. U.S.S.R.* **10**, 267.
Chan-Mou Tchen *Phys. Rev.* To be published.
Gasiorowicz S., Neumann M. and Riddell R. J. Jr. (1956) *Phys. Rev.* **101**, 922.
Kadomtsev B. B. (1958) *Sov. Phys. J.E.T.P.* **6**, 117.
Landau L. (1946) *J. Phys. (U.S.S.R.)* **10**, 25.
Mayer J. E. and Mayer M. G. (1940) *Statistical Mechanics* John Wiley, New York.
Pines D. and Bohm D. (1952) *Phys. Rev.* **85**, 338.
Rosenbluth M. N., MacDonald W. M. and Judd D. L. (1957) *Phys. Rev.* **107**, 1.
Rosenbluth M. N. and Rostoker N. (1959) GAMD-663, Part III—General Atomic Internal Report. Available on request.

OL.
2
961

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 206 to 217. Pergamon Press Ltd. Printed in Northern Ireland

CLASSIFICATION OF DIAGRAMS FOR A PLASMA EQUATION-OF-STATE*

H. B. LEVINE
Lawrence Radiation Laboratory, University of California, Livermore, California

Abstract—Utilizing the MONTROLL-WARD approach to quantum statistics, generalized to many components, we seek an equation of state for a high-temperature low-density plasma. We propose a classification of diagrams analogous to that used by MEERON and by FRIEDMAN in classical statistical mechanics. In the resulting expansion, the leading term is the ideal gas contribution plus the lowest-order exchange contribution plus the ring contribution, the latter representing the effect upon the pressure of collective motions of a completely ionized plasma. The next term appears to represent a contribution due to modifications of free particle motion due to the interaction of the single particles with the rest of the plasma. The third term represents the contribution of two-particle states, both of positive energy between all particle pairs, and negative energy (bound states) between pairs of opposite charge sign. The third term also contains parts representing modifications of the motion of the pair of particles due to interaction of the individual particles with the rest of the plasma. Higher terms represent, successively, contributions of three-, four-, etc., particle states, again including modifications of the single-particle motions. The classification automatically eliminates the ultraviolet catastrophe which ordinarily arises in the treatment of Coulomb force bound states in statistical mechanics, since, in effect, it uses a screened Coulomb potential instead of the ordinary potential. In addition, the short range divergence, which occurs in classical theory because of the 'fall' of the electron to the nucleus, does not arise, being prevented by the uncertainty principle.

1. INTRODUCTION

THE calculation of the equation of state of a plasma, particularly from first principles, is a problem of considerable mathematical complexity. Although the system with which we deal is one of exactly known forces, an advantage not usually encountered in statistical mechanics, in virtually every other respect difficulties arise. First, we must treat a system of at least two components, so that the number of variables we must handle (e.g., the number densities of the components) is large. Even for a system of exactly two components, such as a proton-electron gas, certain aspects of the algebra are considerably complicated relative to, say, an equivalent calculation for the 'one-component' electron gas submerged in a background of uniform positive charge. To a slight extent, but not completely, this difficulty is cancelled by the advantage of the usual constraint of overall electrical neutrality. Second, we must treat a system with long range forces, so that techniques such as cluster and virial expansions are not directly available to us. These series, albeit limited to low densities, constitute almost the only rigorous approach to the evaluation of thermodynamic data by statistical mechanical means. Third, and most serious of all, we must treat a system in which *quantum* mechanics and *quantum* statistics play a predominant role. The difficulty of making precise quantum statistical calculations is well known—indeed until quite recently no one even knew how to *start* such calculations.

As a result of the above and other reasons, previous calculations (LATTER, 1955; COWAN and KIRKWOOD, 1958) have been limited for the most part to techniques involving Thomas-Fermi and Debye-Hückel methods. Although such methods are doubtless reasonable for certain regions of the pressure-density plane, they suffer two serious disadvantages: first, it is extremely difficult to estimate their accuracy at any given temperature and density and, second, there is no logical *ordered* scheme for improving their accuracy in successive steps. Fortunately, a much better approach to the entire problem has now appeared.

It is the purpose of this note to point out that as a consequence of some very recent developments, a rigorous approach to the calculation of plasma equations of state is now plausible. Such calculations in fact have been undertaken, and hopefully will be reported upon at a later date. For the present, we only outline the method whereby such a calculation may be made. We shall limit ourselves for the most part to remarks about high temperature-low density plasmas. It is difficult to give precise numbers for the limits of the temperature-density region we hope can be handled, since these limits will in fact be determined by the rate of convergence of series which we have not yet computed. However, we roughly expect to be able to treat a region describable as follows: The temperature is high enough and the density is low enough so that the system is nearly completely ionized. However, a small number of particles are tied up in two-body bound states, a still smaller number in three-body bound states, etc. If we consider the simplest

* This work was performed under the auspices of the U.S. Atomic Energy Commission.

VOL. 2 1961

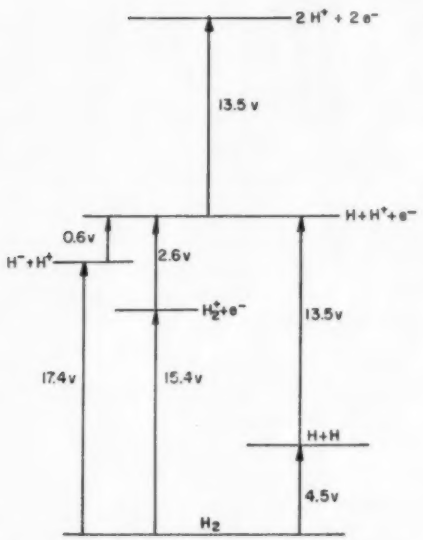


FIG. 1.—Ionization scheme for the H_2 molecule.

possible plasma, i.e. a proton-electron gas, as an example, we can estimate the temperatures and densities we can deal with. In Fig. 1 we have drawn an energy level scheme for the ionization of the H_2 molecule. Excited bound states are omitted for simplicity. We see that at densities typical of those at which the indicated dissociation energy and ionization potentials are observed experimentally (10^{16} to 10^{17} molecules of H_2/cm^3) and temperatures of about 31·5 volts ($\cong 3{\cdot}7 \times 10^5$ °K) the system is certainly completely ionized. We have here ignored the high degeneracy of the 'ionized states' as opposed to the bound states. Because of this degeneracy, ionization is in fact complete at much lower temperatures. A recent calculation (HARRIS, private communication) tends to indicate that ionization is complete at densities as high as 10^{19} to 10^{20} molecules of H_2/cm^3 (room temperature gas density $\cong 3 \times 10^{19}$ molecules of H_2/cm^3) and temperatures as low as 5 volts ($\cong 6 \times 10^4$ °K), and may also be complete at still higher densities at this temperature. (It is of course always complete at very high densities, i.e. many times solid densities, at any temperature. However, in this region of *pressure* ionization, as distinct from the *temperature* ionization mentioned above, we probably cannot describe the unionized part of the system in terms of two-body bound states, three-body bound states, etc. and our approach correspondingly may not be applicable.) For plasmas containing heavy nuclei, the energy required for ionization is much greater than that required for the proton-electron case, and our approach will therefore be useful, e.g., only at temperatures $\gg 5$ volts for densities of 10^{19} to 10^{20} nuclei/cm^3. Thus, we may in fact find that we are in practice limited to calculations for very light elements only.

In Section 2 we discuss the MONTROLL-WARD (MONTROLL and WARD, 1958) approach to equilibrium quantum statistical mechanics. In Section 3 we discuss the recent work of MEERON (1957, 1958) and FRIEDMAN (1959) on higher order terms in the classical Debye-Hückel theory. In Section 4 we combine the ideas of MONTROLL and WARD (1958) and MEERON (1957, 1958) and FRIEDMAN (1959) to produce an expansion for the equation of state of a quantum mechanical plasma. We also indicate the physical interpretation of the first few terms. In Section 5, we make some concluding comments.

2. THE MONTROLL-WARD APPROACH TO QUANTUM STATISTICS

MONTROLL and WARD (1958) have recently presented an elegant approach to the evaluation of the pressure of a quantum mechanical assembly, via the method of the Grand Canonical ensemble. Their method may be described as a quantum mechanical generalization of the classical cluster integral method of MAYER (MAYER and MAYER, 1940). In their method one evaluates the pressure as a sum over an infinite set of integrals. Corresponding to each integral there is a certain graph which determines the integrand. For Coulomb forces the individual integrals mostly diverge. However, the graphs (i.e. the integrals corresponding to the graphs) may be collected into an infinite collection of subsets, such that the (infinite) sum of graphs in a given subset sums to a finite quantity. This finite quantity then represents the contribution to the pressure from the graphs of the given subset. For the one component electron gas submerged in a uniform background of positive charge (the density of the latter being chosen so as to make the entire system neutral), MONTROLL and WARD (1958) have computed the contribution to the pressure from the so-called ring graphs. In order to describe suitably these graphs, we indicate rules for their construction, generalized to multicomponent systems (LEVINE, 1960).

We define a free particle propagator,

$$L_i(\mathbf{R}_2 - \mathbf{R}_1, \beta_2 - \beta_1),$$

by

$$L_i(\mathbf{R}_2 - \mathbf{R}_1, \beta_2 - \beta_1)$$

$$= \left[\frac{2\pi m_i}{\hbar^2(\beta_2 - \beta_1)}\right]^{3/2} \exp\left\{-\frac{m_i}{2\hbar^2}\frac{(\mathbf{R}_2 - \mathbf{R}_1)^2}{\beta_2 - \beta_1}\right\} \delta(\sigma_2, \sigma_1)$$

$$\text{if } \beta_2 \geqslant \beta_1;$$

$$= 0 \text{ if } \beta_2 < \beta_1. \qquad (1)$$

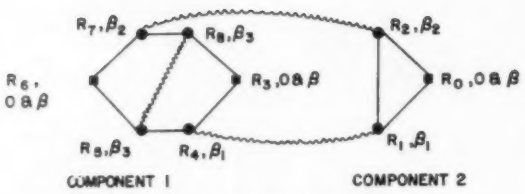


FIG. 2.—A typical quantum diagram.

We have one such function for each component, i, in the system. (By component we mean, e.g., electrons, protons, etc). The vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ are position vectors of a particle, the scalars σ_1 and σ_2 are spin co-ordinates of the same particle, the quantities β_1 and β_2 are reciprocal temperature variables ($\beta = 1/kT$), the quantity m_i is the mass of a particle of species i, and the quantities h and $\hbar$ have their usual meaning. The function $\delta(\sigma_2, \sigma_1)$ is the Kronecker delta. We represent this propagator, L_i, by a straight line. We also have a pair potential,

$$U(\mathbf{R}_2 - \mathbf{R}_1) = C_1 C_2 e^2 (8\pi^3)^{-1/2} \int\int\int_{-\infty}^{\infty} u(k) \exp\left[-i\mathbf{k}\cdot(\mathbf{R}_2 - \mathbf{R}_1)\right] d\mathbf{k} \qquad (k = |\mathbf{k}|), \tag{2}$$

which we represent by a wavy line. In equation (2), C_1 and C_2 are the charge numbers of the two particles involved ($C = -1$ for electrons, $C = +1$ for protons, etc.), $-e$ is the charge on the electron and

$$u(k) = (2/\pi)^{\frac{1}{2}}(1/k^2) \tag{3}$$

is the Fourier transform of the Coulomb potential.* A graph is then made, as in Fig. 2, by connecting vertices, indicated by circular dots, into rings via free particle propagators. In a given ring all propagators, L_i, are of the same component i. The rings are termed *torons*. The torons are then connected together via wavy lines. Wavy lines may also connect two points on the same toron (we then speak of this line as an exchange interaction line). In addition to the circular vertices, each toron has one or more square vertices. The number of these square vertices is termed the *sectionality*† of the toron. Each circular vertex is the termination of two free particle propagators and one interaction line. Each square vertex is the termination of two free particle propagators only. An exchange interaction is allowed only on torons of sectionality $\geqslant 2$, and then only if the two circular vertices are 'separated' on the toron by having a square vertex between them on either side of the toron. Subject to certain restrictions (LEVINE, 1960) the pressure then is obtained as the sum of all connected graphs. To evaluate the integral for a given graph we assign a co-ordinate variable $\mathbf{R}_j$ to each vertex, a reciprocal temperature variable, β_j, to each pair of circular vertices connected by a wavy line and two reciprocal temperatures, zero and β (= actual reciprocal temperature of the system) to each square vertex. We then integrate all $\mathbf{R}_j$'s over the entire volume V of the system, integrate all β_j's in a chain-wise Laplace-Faltung fashion (MONTROLL and WARD, 1958; LEVINE, 1960) from 0 to β, and multiply by

$$V^{-1} \prod_i \frac{\zeta_i^{t_i} S_i^{t_i}}{t_i!}, \tag{4}$$

where

$$S_i = (2s_i + 1)\gamma_i \tag{5a}$$

$$\zeta_i = \gamma_i z_i \tag{5b}$$

$$\gamma_i = \begin{cases} -1 \text{ if component } i \text{ obeys Fermi-Dirac Statistics} \\ +1 \text{ if component } i \text{ obeys Bose-Einstein Statistics.} \end{cases} \tag{5c}$$

In equations (4) and (5), t_i is the number of square vertices of component i in the diagram, $2s_i + 1$ is the spin degeneracy of a free-particle eigenstate for a particle of component i ($2s_i + 1 = 2$ for electrons, 2 for protons, 1 for a ^{4}He nucleus, etc.), and $z_i = \exp \beta\mu_i$ is the absolute activity of species i (μ_i = chemical potential of species i in units of ergs/particle). For example, the integral for Fig. 2 is

$$\int\cdots\int d\mathbf{R}_0 \cdots d\mathbf{R}_8 \int_0^{\beta}\int_0^{\beta_3}\int_0^{\beta_2} d\beta_1\, d\beta_2\, d\beta_3$$
$$\times\, U(\mathbf{R}_4 - \mathbf{R}_1) U(\mathbf{R}_8 - \mathbf{R}_5) U(\mathbf{R}_7 - \mathbf{R}_2)$$
$$\times\, L_1(\mathbf{R}_4 - \mathbf{R}_3, \beta_1) L_1(\mathbf{R}_5 - \mathbf{R}_4, \beta_3 - \beta_1) L_1(\mathbf{R}_6 - \mathbf{R}_5, \beta - \beta_3)$$
$$\times\, L_1(\mathbf{R}_7 - \mathbf{R}_6, \beta_2) L_1(\mathbf{R}_8 - \mathbf{R}_7, \beta_3 - \beta_2) L_1(\mathbf{R}_3 - \mathbf{R}_8, \beta - \beta_3)$$
$$\times\, L_2(\mathbf{R}_1 - \mathbf{R}_0, \beta_1) L_2(\mathbf{R}_2 - \mathbf{R}_1, \beta_2 - \beta_1) L_2(\mathbf{R}_0 - \mathbf{R}_2, \beta - \beta_2)$$
$$\times\, V^{-1} \frac{\zeta_1^2}{2!} \frac{\zeta_2}{1!} S_1 S_2. \tag{6}$$

* We do not consider the possibility of external fields, or spin interactions, although the theory can in principle be modified to do so. We also do not consider Lorentz forces. The latter can only be important at temperatures high enough so that the electrons are relativistic. For low densities, the temperatures necessary to produce relativistic electrons can be estimated by computing the mean electron velocity by setting $m_{\text{electron}} v^2 = kT$ and computing v^2/c^2. The electrons only begin to be relativistic at about 10^5 volts ($\cong 10^9$ °K). Thus our approach will also be invalid at these high temperatures and in fact this upper temperature limit will decrease with increasing density, since increasing density will increase the mean electron velocity. However, we still expect to be left with a broad temperature band (perhaps 1 volt to 10^5 volts) for which our approach is valid even at densities as high as 10^{20} particles/cm^3 (and possibly higher).

† MONTROLL and WARD (1958) use instead the term *order*.

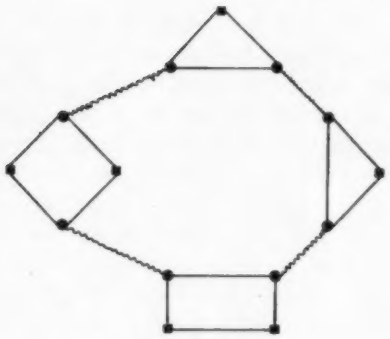


FIG. 3.—A ring diagram.

The ring graphs consist of rings of two or more torons, connected to each other by single interaction lines, with no exchange interactions in any toron. For such graphs, a typical one of which is pictured in Fig. 3, the integrations over the co-ordinates $\mathbf{R}_j$ can be performed by repeated application of the Faltung theorem for Fourier transforms. In addition, the β_j integrations can be reduced to one infinite series if one considers the sum over all rings containing a fixed number of torons (MONTROLL and WARD, 1958). By then summing over all rings of two or more torons, one obtains the ring contribution to the pressure

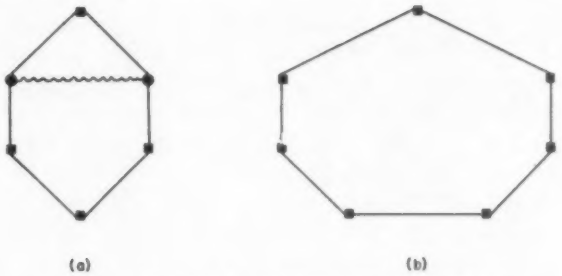
$$4\pi^2\beta P_{\text{ring}} = \sum_{j=-\infty}^{\infty} \int_0^{\infty} k^2\, dk\{\lambda_j(k)u(k) - \ln[1 + \lambda_j(k)u(k)]\}, \qquad (7)$$

where

$$\lambda_j(k) = -(8\pi^3)^{1/2}\beta e^2 \sum_i \left(\frac{2\pi m_i}{h^2\beta}\right)^{3/2} C_i^2 S_i \times \sum_{t=1}^{\infty} \frac{\zeta_i^t}{t^{3/2}} \int_0^1 \exp\left[-2\pi i j t x - \frac{t\beta\hbar^2 k^2}{2m_i}\, x(1-x)\right] dx. \qquad (8)$$

These equations represent the multicomponent generalization of the result of MONTROLL and WARD (1958), the sum over i in equation (8) being over all components. Although these equations still constitute a fairly intractable form, they nevertheless represent a considerable improvement over the starting point. We shall in fact for the present regard equations (7) and (8) as an illustration of the degree of simplification at which we think of the integrals as being 'solved'. Reduction of other integrals to forms of equivalent complexity will then constitute 'evaluation' of these integrals. Only at a later stage can we consider additional simplification

In obtaining equations (7) and (8), we have failed to consider one very special type of ring, viz. the ring of *one* toron.* Such rings are more properly termed



(a) (b)

FIG. 4.—(a) A ring of one toron (exchange diagram). (b) An ideal gas diagram.

'first-order exchange graphs,' since the single interaction is of necessity an exchange interaction. We have pictured a typical graph of this type in Fig. 4(a). Utilizing methods similar to those used for evaluating the rings, these graphs are easily evaluated. Summing over all such graphs, we find a contribution to the pressure of

$$\beta P_{\text{exchange}} = -\frac{4\pi e^2}{h^4\beta} \sum_i m_i^2 C_i^2 S_i \times \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{\zeta_i^{p+q}}{p+q} \cdot \frac{1}{\sqrt{pq}} = -\frac{4\pi^2 e^2}{h^4} \sum_i m_i^2 C_i^2 S_i \int_0^{\infty} [I_{-1/2}(\zeta_i e^{-\beta x})]^2\, dx, \qquad (9)$$

where

$$I_{-1/2}(\xi) = -\int_0^{\infty} \frac{\xi y^2 e^{-y^2}\, dy}{(1 - \xi e^{-y^2})^2}. \qquad (10)$$

In equation (9), the sum over i is again a sum over all components.

Another important group of graphs are those consisting of a single toron, of sectionality $\geqslant 1$, one of which is shown in Fig. 4(b). These graphs contribute the ideal gas part of the pressure, i.e. the only term which does not vanish as $e \to 0$. They are easily evaluated (MONTROLL and WARD, 1958) and give a contribution

$$\beta P_{\text{ideal}} = -\frac{4}{\sqrt{\pi}} \sum_i \left(\frac{2\pi m_i}{h^2\beta}\right)^{3/2} S_i \times \int_0^{\infty} x^2 \ln[1 - \zeta_i e^{-x^2}]\, dx, \qquad (11)$$

the sum again being over all constituents.

In the diagram classification which follows, the three contributions, equations (7), (9) and (11) shall together constitute the first term. By themselves, they constitute a first approximation to the equation of state of a plasma in two limits (MONTROLL and WARD, 1958). Thus, at high temperatures and low densities, particularly as $h \to 0$, they lead to a Debye-Hückel equation of state for a classical system of charged particles, while at low temperatures and high densities,

* The author is indebted to Prof. Z. W. SALSBURG for pointing out the importance of these graphs to him.

they lead, after appropriate thermodynamic manipulation, to the GELL-MANN–BRUECKNER expression for the ground state energy of an electron gas (GELL-MANN and BRUECKNER, 1957). However, the classification we shall make will stress particularly the Debye-Hückel, rather than the GELL-MANN–BRUECKNER limit. In order to develop this classification, we first must briefly review some recent work by MEERON (1957, 1958) and FRIEDMAN (1959).

3. HIGH-ORDER CLASSICAL IONIC SOLUTION THEORY

MEERON (1957, 1958) and FRIEDMAN (1959) have extended MAYER'S (1950) approach to the classical Debye-Hückel theory of ionic solutions. The essential feature of this method is the classification of the infinite set of graphs which determine the osmotic pressure of an electrolytic solution into an infinite collection of subsets, each of which gives a finite contribution to the pressure. The collections are so ordered as to give a low density MEERON (1957, 1958) and FRIEDMAN (1959) (and hence, in classical statistical mechanics, directly also a high temperature) expansion for the osmotic pressure. For our purposes, we may take the solvent to be a vacuum of dielectric constant unity, and also ignore the many-body interaction terms which FRIEDMAN considers, since the latter arise only because of the detailed structure of the solvent molecules and ions. Both authors also assume strong short-range repulsions between the ions and, indeed, if these forces are omitted, for ions of opposite charge, divergences arise which would vitiate the theory. As we shall point out in the next section, no such divergences should appear in a quantum mechanical plasma, despite the necessary absence of short-range repulsions between nuclei and electrons in our Hamiltonian. In order to deal in parallel fashion with both the classical and quantum cases, it is convenient for the most part to leave these repulsions out of our remarks about classical theory, only interjecting them when necessary to save the day, as it were. Our Hamiltonian and their Hamiltonian are then identical, our positive ions being nuclei, and our negative ions electrons. Finally, of course, their expression for osmotic pressure becomes an expression for mechanical pressure.

Since their theory is classical, the graphs involved are slightly different from those in the previous section. Instead, they consist simply of vertices of a single type, which we draw as round dots, connected to each other by interaction (wavy) lines only. A few typical diagrams are shown in Fig. 5(a) to 5(e). Corresponding to each vertex there is a position co-ordinate $\mathbf{R}_i$, which represents the position of one particle of any component. A wavy line between two co-ordinates $\mathbf{R}_i$ and $\mathbf{R}_j$ represents $U(\mathbf{R}_i - \mathbf{R}_j)$, where $U(\mathbf{R}_i - \mathbf{R}_j)$ is the pair potential, equation (2), with charge numbers C_i and C_j corresponding to the species occupying positions $\mathbf{R}_i$ and $\mathbf{R}_j$. The integral for a

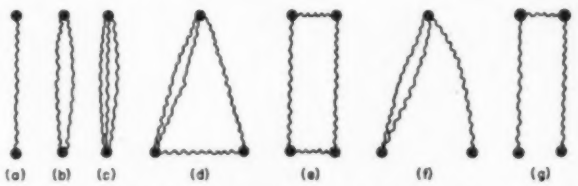


FIG. 5.—(a) to (e) Irreducible classical diagrams. (f), (g) Reducible classical diagrams.

given graph is simply the integral of all the $\mathbf{R}_i$ vectors in the graph over the entire volume V, multiplied by a factor

$$V^{-1} \prod_i \frac{\rho_i^{t_i}}{t_i!} \prod_i \prod_j \frac{(-\beta)^{k_{ij}}}{k_{ij}!}. \tag{12}$$

In expression (12), ρ_i is the *mean number density* of component i, t_i is the number of particles (i.e. vertices) of component i in the particular graph, and the single product is over all components in the system. The double product is over all *pairs of vertices* in the diagram, k_{ij} being the number of interaction lines directly connecting vertices i and j. For example, the integral for Fig. 5(d) is

$$V^{-1} \frac{\rho_1^3}{3!} \frac{\beta^4}{2!} \int\int\int d\mathbf{R}_1\, d\mathbf{R}_2\, d\mathbf{R}_3 [U(\mathbf{R}_1 - \mathbf{R}_2)]^2 \times U(\mathbf{R}_2 - \mathbf{R}_3) U(\mathbf{R}_1 - \mathbf{R}_3), \tag{13}$$

with all charge numbers being C_1, in the case where all three vertices are occupied by component 1. As another example, if the top vertex of this graph were of component 2 and the other two vertices of component 1, the integral would be

$$V^{-1} \frac{\rho_1^2 \rho_2}{2!} \frac{\beta^4}{2!} \int\int\int d\mathbf{R}_1\, d\mathbf{R}_2\, d\mathbf{R}_3 [U(\mathbf{R}_1 - \mathbf{R}_2)]^2 \times U(\mathbf{R}_2 - \mathbf{R}_3) U(\mathbf{R}_1 - \mathbf{R}_3), \tag{14}$$

and the charge numbers would be C_1C_2 in $U(\mathbf{R}_1 - \mathbf{R}_2)$, C_1C_2 in $U(\mathbf{R}_2 - \mathbf{R}_3)$, and C_1^2 in $U(\mathbf{R}_1 - \mathbf{R}_3)$.

To evaluate the pressure using these graphs, we first must take the sum, $\mathcal{S}$, over all *irreducible* graphs. This collection of graphs consists of all those connected graphs with the property that they are not broken into two or more separate pieces when any one vertex is cut out of the graph (MAYER and MAYER, 1940). The graphs of Figs. 5(a) to 5(e) are irreducible. Graphs which contain one or more vertices the removal of which breaks them into two or more pieces

are termed *reducible*. In Figs. 5(f) and 5(g) we show two reducible graphs. The pressure is related to the sum, $\mathscr{S}$, by (MAYER, 1950)

$$P\beta = \sum_i \rho_i + \mathscr{S} - \sum_i \rho_i \left(\frac{\partial \mathscr{S}}{\partial \rho_i}\right)_{\beta,\ \text{all other } \rho_j\text{'s.}} \tag{15}$$

The first sum on the right side of this equation is the ideal gas contribution to the pressure.

Although the integrals for the individual diagrams mostly diverge, they may be grouped into summable subsets. The first such subset are the graphs of the type of Fig. 5(a). The integrals for these graphs are

$$-V^{-1}\frac{\rho_i^2}{2!}\beta\int\int d\mathbf{R}_1\, d\mathbf{R}_2 C_i^2 e^2 \frac{1}{|\mathbf{R}_1 - \mathbf{R}_2|}, \tag{16a}$$

or

$$-V^{-1}\rho_i\rho_j\beta\int\int d\mathbf{R}_1\, d\mathbf{R}_2 C_i C_j e^2 \frac{1}{|\mathbf{R}_1 - \mathbf{R}_2|}, \tag{16b}$$

according as both vertices are or are not occupied by the same component. Summing over all such diagrams (i.e. over all component pairs), one obtains, as $V \to \infty$,

$$-2\pi\beta e^2\left[\sum_i \rho_i C_i\right]^2 \int_0^\infty R\, dR \tag{17}$$

as the contribution to the pressure of these diagrams. Despite the divergence of the integral, we may take (17) to be equal to zero, since the sum $\sum_i \rho_i C_i$ vanishes for a system which is electrically neutral overall (that $0^2 \cdot \infty = 0$ in this case can be rigorously established by more careful treatment of the pertinent limiting processes).

The remaining diagrams are classified as follows: A given vertex is said to be a *node of order n* if n interaction lines terminate on it. A node of order $\geqslant 3$ is said to be a *branch point*. Except for the graphs of Fig. 5(a), which sum to zero, only nodes of order $\geqslant 2$ occur. The diagrams are then grouped according to the number of branch points which appear in the diagram. The leading term is then the sum, $\mathscr{S}_0$, of all simple ring graphs, such as Figs. 5(b) and 5(e), since these diagrams contain no branch points. This sum was first evaluated by MAYER (1950) using the Faltung theorem for Fourier transforms, and leads to a Debye-Hückel pressure term.

The next terms in the expansion consist, successively, of the sums of all graphs containing two, three, four, etc., branch points. There is no term with only one branch point, since any such diagram is necessarily reducible. In dealing with these terms one first considers an *initial graph* which does not contain any nodes of order two, for example the graph in Fig. 5(c). One then 'generates' additional graphs which do contain nodes of order two by successively replacing the individual interaction line of the initial graph by chains of nodes of order two. In Fig. 6 we have pictured some of the graphs generated by the graph of Fig. 5(c) (we formally define the generation process so that the initial graph generates itself).

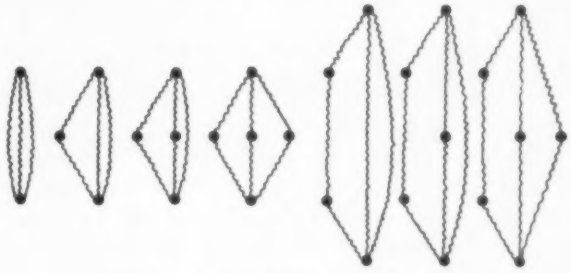


FIG. 6.—Diagrams generated by Fig. 5(c).

Integration over the co-ordinates of any vertex on a chain is easily done. By summing these integrals over all chain lengths (from zero intermediate vertices to an infinite number of intermediate vertices) for all bonds in the initial graph, one is led to a simple expression for the sum of the generated set of graphs. This expression is formally identical to the original expression for the initial graph, as discussed above, except that the original Coulomb potentials are replaced by screened Coulomb potentials, i.e.

$$U(\mathbf{R}_2 - \mathbf{R}_1) = C_1 C_2 e^2/|\mathbf{R}_2 - \mathbf{R}_1| \tag{18a}$$

is replaced by

$$U(\mathbf{R}_2 - \mathbf{R}_1) = C_1 C_2 e^2 \frac{\exp(-\alpha\,|\,\mathbf{R}_2 - \mathbf{R}_1|)}{|\mathbf{R}_2 - \mathbf{R}_1|}, \tag{18b}$$

where

$$\alpha^2 = 4\pi\beta e^2 \sum_i \rho_i C_i^2\ ; \tag{19}$$

or, alternatively,

$$u(k) = (2/\pi)^{1/2}(1/k^2) \tag{20a}$$

is replaced by

$$u(k) = (2/\pi)^{1/2}\frac{1}{k^2 + \alpha^2}. \tag{20b}$$

Since the term following the Debye-Hückel pressure term is defined as the sum over all graphs containing two branch points, it is given by the sum over all graphs generated by the initial graphs shown in Fig. 7. This summation is easily performed and leads to the following expression for the term, $\mathscr{S}_2$, in question:

$$\mathscr{S}_2 = 2\pi\sum_i \rho_i^2 \int_0^\infty R^2\{e^{q_{ii}} - 1 - q_{ii} - \tfrac{1}{2}q_{ii}^2\}\, dR + 4\pi\sum_{i<j}\sum \rho_i\rho_j \int_0^\infty R^2\{e^{q_{ij}} - 1 - q_{ij} - \tfrac{1}{2}q_{ij}^2\} dR, \tag{21}$$

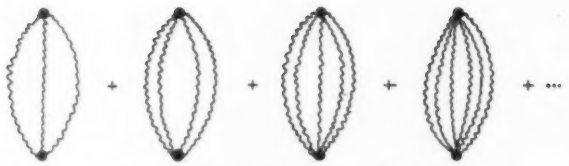


FIG. 7.—Initial graphs contributing to $\mathscr{S}_2$.

where

$$q_{ij} \equiv q_{ij}(R) = -\beta e^2 C_i C_j R^{-1} \exp(-\alpha R). \tag{22}$$

The single sum in equation (21) represents the contribution of all diagrams with two branch points, both of which are of the same component, the sum being over all components. The double sum represents the corresponding contribution from diagrams in which the branch points are of different components, the double sum being over all pairs of unlike components. All terms in the first sum are finite, as are all terms in the second sum for which $C_i C_j > 0$. However, for terms for which $C_i C_j \ll 0$, the integrals diverge because of an essential singularity in the integrands at $R \to 0$. In the classical theory this divergence is eliminated by bringing in the short-range repulsions between ions. Thus, if one assumes a hard-sphere repulsion, the lower limits, 0, in each integral are changed to $A_{ij} > 0$, the distance of closest approach of a pair of ions, one of species i and one of species j, and the integral is then finite.

For purposes of discussion, it is convenient to write equation (21) in the form

$$\begin{aligned}\mathscr{S}_2 = {} & 2\pi \sum_i \sum_j \rho_i \rho_j \int_0^\infty R^2 \{e^{q_{ij}} - 1\}\, dR \\ & -2\pi \sum_i \sum_j \rho_i \rho_j \int_0^\infty R^2 q_{ij}\, dR \\ & -2\pi \sum_i \sum_j \rho_i \rho_j \int_0^\infty R^2 q_{ij}^2\, dR,\end{aligned} \tag{23}$$

where the two sums, over i and j, in each term run independently over all components. The second of the three terms on the right side of equation (23) vanishes identically, since substitution of equation (22) for q_{ij} shows it to have a coefficient containing the factor $\sum_i \rho_i C_i$. The third term is finite, and is easily computed to be $-\alpha^3/32\pi$. The value of the Debye-Hückel term, $\mathscr{S}_0$, computed from the ring diagrams by MAYER (1950) is $\alpha^3/12\pi$. Hence $\mathscr{S}_2$ can be thought of as having two parts, one of the order of the Debye-Hückel term, and the other, as represented by the first term of equation (23), in the form of a second virial coefficient for a pair of particles interacting via the screened Coulomb potential, equation (18b). This first integral thus represents the effect upon the pressure of two-body interactions.* We emphasize that just as the ordinary second virial coefficient contains contributions from both positive energy (scattering) and negative energy (bound) states, so our modified second virial coefficient also contains contributions from both types of states. Furthermore, the physical meaning of the divergence as $R \to 0$ is clear. This divergence, which is entirely contained in the second virial coefficient part of equation (23), arises from the strong $-1/R$ attraction between particles of unlike charge at small R—an attraction so strong that it causes the two particle to 'fall' to zero separation despite the low (in fact zero!) statistical weight of such a state. Thus the divergence represents just one more manifestation of the familiar fact that classical mechanics does a very poor job of representing bound states.

Although we have divided $\mathscr{S}_2$ into three parts in equation (23) to clarify its physical meaning, such a division is undesirable from a mathematical point of view. Thus, the fact that one of the integrals in equation (23) is of order α^3 might lead us falsely to conclude that $\mathscr{S}_2$ itself was of this order. In fact, the second virial coefficient part of equation (23) also contains a part of order α^3, which is identical with and of opposite sign to the term $-\alpha^3/32\pi$ (this is obvious from equation (21)). These terms cancel, the order in α of $\mathscr{S}_2$ then being > 3.

We see that the essential character of the term $\mathscr{S}_2$ is that it represents the effect of two-body interactions via a potential modified at long distances because of the presence of additional particles. In an exactly similar way, it can be shown that the term, $\mathscr{S}_3$, obtained from all diagrams containing three branch points, represents the effect of three-body interactions via the same modified potential. More generally, the term $\mathscr{S}_n$, obtained from all diagrams containing n branch points, represents the effect of n-body interactions via the modified potential. Of even greater importance, FRIEDMAN has shown (1959), although not with complete mathematical rigour, that the expansion represented by the series

$$\mathscr{S} = \mathscr{S}_0 + \sum_{n=2}^{\infty} \mathscr{S}_n \tag{24}$$

is a low-density, high-temperature expansion. This, of course, is an essential feature of any expansion having the physical interpretation we have indicated.

* At short distances, the ordinary and screened Coulomb potentials are essentially equal, and the particles interact in their 'normal' fashion. However at large distances, the charges on the two particles polarize the intervening medium and their interaction is thereby modified. The screening factor, $\exp(-\alpha R)$, is the mathematical manifestation of this polarization.

In the next section, we shall generalize the classification of diagrams described here so as to apply it to the quantum-statistical diagrams of MONTROLL and WARD (1958).

4. QUANTUM-MECHANICAL LOW DENSITY HIGH TEMPERATURE EXPANSIONS

It is the purpose of this section to apply the ordering scheme of MEERON (1957, 1958) and FRIEDMAN (1959) to the graphs of MONTROLL and WARD (1958). Since we have not made any detailed calculations along these lines, we can only indicate in a general way the nature of the attack. We mention points of difficulty we foresee, but there may be points of difficulty we do not foresee.

The first thing one must do is establish the relation between the classical and quantum graphs. MONTROLL and WARD (1958) have essentially stated this relation, although they did not provide a formal proof. Stated simply, it is: each toron in a quantum diagram corresponds to a single vertex in a classical diagram. An exact mathematical demonstration of this can be provided (LEVINE, 1960), although we do not give it here. The essential feature is the observation that the free particle propagator, equation (1), goes over in the classical limit, $h \to 0$, to the Dirac delta function $\delta(\mathbf{R}_2 - \mathbf{R}_1)$ (for $\beta_2 \geqslant \beta_1$). As a result, the complex integrand for a quantum diagram, which involves the co-ordinates for all vertices in each toron, vanishes except when all vertex positions of a given toron are the same, i.e. when the toron corresponds to a single vertex. In addition, the quantum β-integrations become separable when $h \to 0$, and lead simply to the factors $(-\beta)^{k_{ij}}/k_{ij}!$ of expression (12).

By analogy with the classical diagrams, we therefore call any toron, whatever its sectionality, a *toron of nodal order n* if n interaction lines terminate upon it. With this definition, an exchange interaction adds two to the nodal order of a toron since both ends of an exchange interaction terminate upon the same toron. We similarly term any toron of nodal order $\geqslant 3$ a branch point. We propose to classify quantum diagrams by the number of branch points just as was done in classical statistics. One difficulty prevents straightforward application of this scheme, and we must discuss this next.

In the classical case, except for Fig. 5(a), which contributes zero to $\mathscr{S}$ no diagrams appear which contain nodes of order one. This is a direct consequence of the fact that the ρ_i's were used as independent variables, and therefore the entire theory could be formulated in terms of irreducible graphs only. In the quantum case, however, graphs containing torons of nodal order do appear. This is because the z_i's are used as independent variables, and the rules therefore require that we sum over all graphs (MONTROLL and WARD, 1958; LEVINE, 1960), reducible as well as irreducible.* The former group introduces an infinity of graphs containing nodes of order one.† Any graph containing nodes of order one formally integrates to infinity, since its value may be shown to be proportional to a positive integral power of $u(0)$. (From equation (3), we have $u(0) = \infty$.) For example, consider the sum over all graphs consisting of two torons, connected by a single interaction line, with no exchange interactions. These graphs are the quantum generalization of the single classical diagram of Fig. 5(a). An example of such a graph is shown in Fig. 8(a). Their evaluation is quite straightforward, and one finds for the sum of such graphs the expression

$$-2\pi\beta e^2\left[\sum_i \left(\frac{2\pi m_i}{h^2\beta}\right)^{3/2} C_i S_i \sum_{t=1}^{\infty} \frac{\zeta_i{}^t}{t^{3/2}}\right]^2 \left(\frac{\pi}{2}\right)^{1/2} u(0)$$

$$= -2\pi\beta e^2\left[\sum_i \left(\frac{2\pi m_i}{h^2\beta}\right)^{3/2} C_i S_i \sum_{t=1}^{\infty} \frac{\zeta_i{}^t}{t^{3/2}}\right]^2 \int_0^{\infty} R\, dR. \tag{25}$$

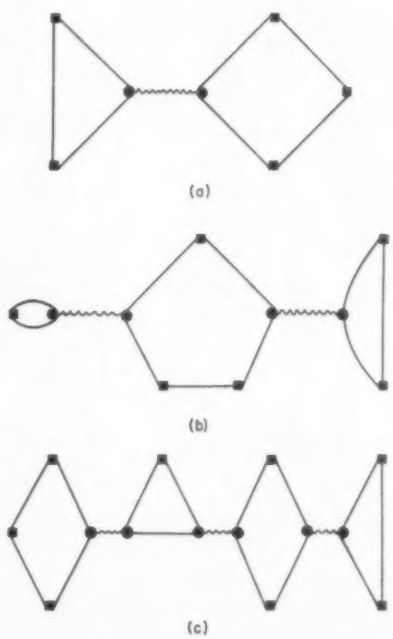


FIG. 8.—Quantum chain diagrams.

* In the quantum case, we use the words *irreducible* and *reducible* in a manner analogous to the classical case. Thus a quantum graph is reducible (irreducible) if there is (is not) one or more torons, removal of which breaks the graph into two or more pieces.

† It is worth remarking in this connexion that we would *prefer* to use the ρ_i's as independent variables in the quantum case also, since we seek, after all, a pressure-density-temperature relationship. Furthermore, the electrical neutrality condition would be most easily applied in a density rather than an activity, expansion. Unfortunately, however, there does not exist at present any analysis of the pressure in terms of density and temperature for quantum statistics.

Expression (25) is the quantum analogue of the classical expression (17). In the limit of a free particle system $e \to 0$, the density, ρ_i^0, of component i, is given by

$$\rho_i^0 = \left(\frac{2\pi m_i}{h^2\beta}\right)^{3/2} S_i \sum_{t=1}^{\infty} \frac{\zeta_i^t}{t^{3/2}}, \tag{26}$$

so that expression (25) may be written as

$$-2\pi\beta e^2 \left[\sum_i \rho_i^0 C_i\right]^2 \int_0^\infty R\, dR. \tag{26a}$$

We now propose to substitute for the normal electrical neutrality condition, $\sum_i \rho_i C_i = 0$, a new condition

$$\sum_i \rho_i^0 C_i = 0. \tag{27}$$

It is apparent (although we have not rigorously proved it), that the value of the integral for any graph containing $m \geqslant 1$ torons of nodal order one is proportional to

$$\left[\sum_i \rho_i^0 C_i\right]^m [u(0)]^n. \quad (n \text{ a positive integer}) \tag{28}$$

We expect that careful treatment of the pertinent limiting processes will show that $0^m \cdot \infty^n = 0$ for all such graphs. In fact, if we first sum over all length toron chains, and then integrate, we expect to introduce a screened potential, for which $u(0)$ is finite, in a manner similar to that which was discussed in the previous sections. For example consider the diagrams of Figs, 8(a), 8(b), 8(c). These represent typical chains of length two, three, four torons. The sum over all graphs consisting of a single chain of length $\geqslant 2$ torons is easily computed. The value of this sum is

$$-\frac{1}{2}\frac{\left[\sum_i \rho_i^0 C_i\right]^2}{\sum_i \rho_i^0 C_i^2} = -2\pi\beta e^2 \left[\sum_i \rho_i^0 C_i\right]^2 u'(0), \tag{29}$$

where

$$u'(k) = (2/\pi)^{1/2} \frac{1}{k^2 + \alpha'^2} \tag{30}$$

is the Fourier transform of the screened Coulomb potential, the screening constant α' having the classical value, equation (19), except that each ρ_i of equation (19) is replaced by ρ_i^0.

If one accepts the new neutrality condition, equation (27), all graphs containing torons of nodal order one may be ignored, insofar as obtaining an expression for the pressure in terms of β and the z_i's is concerned. *Unfortunately, it is not clear that all such graphs may be ignored in obtaining expressions for the densities, ρ_i,* in terms of β and the z_i's. This is because one differentiates ρ to get the ρ_i's and derivatives such as

$$z_i \frac{\partial}{\partial z_i}\left[\sum_i \rho_i^0 C_i\right]^m \left[u(0)\right]^n \tag{31}$$

enter (see expression (28)). The expression (31) does not contain, for $m = 1$, the factor $\sum \rho_i^0 C_i$. Thus, there is no trivial argument that the condition, equation (27), implies that graphs containing only one

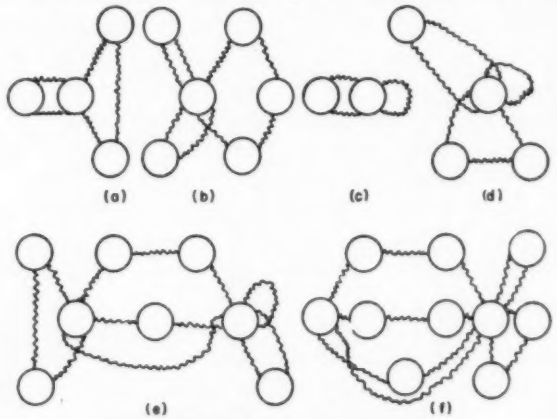


FIG. 9.—Reducible quantum diagrams.

toron of nodal order one may be omitted, although graphs with two or more torons of nodal order one ($m \geqslant 2$) can be omitted. At the present time this presents the one difficulty which may hinder application of the MEERON-FRIEDMAN classification scheme, since diagrams containing torons of nodal order one cannot be classified by their method. We shall, however, in the remainder of this paper assume that any graph containing one or more torons of nodal order one may be ignored completely.

Using this hypothesis, the remaining graphs fall into two groups. The first group consists of the irreducible graphs, which we classify by the MEERON-FRIEDMAN scheme. The second group consists of reducible graphs which do not contain torons of nodal order one. A few examples of such graphs are shown in Fig. 9(a) to (f). To simplify the figure we have altered our method of drawing somewhat. We now use the circle to denote a toron, whose sectionality we do not explicitly show. The wavy lines still denote interactions. If a wavy line connects a toron to itself, it is understood that the toron so connected is of sectionality $\geqslant 2$. For all other torons the sectionality is understood to be $\geqslant 1$. We also classify the graphs of this new group by the MEERON-FRIEDMAN scheme.

The first term in our expansion thus is the sum of the ideal gas graphs, the rings of one toron (first order exchange graphs), and the rings of MONTROLL and WARD, since these graphs all have no branch points.

Their contribution, P_0, to the pressure is the sum of equations (7), (9), and (11). Thus P_0 is the sum of the ideal gas pressure, a part due to simple exchange interactions, and a part due to collective oscillations in the plasma.

The second term in our expansion, P_1, is the sum of all graphs containing one branch point. Since graphs

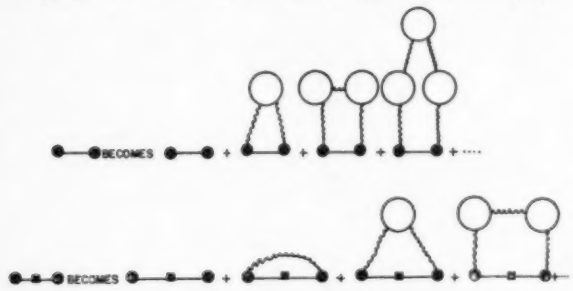


FIG. 10.—Modification of the free particle propagator by the other particles of the system.

with one branch point are all reducible, there is no corresponding term in the classical density expansion, although a classical activity expansion would have such a term. We have pictured some of these graphs in Figs. 9(a) to 9(d). We think of the propagators which make up the branch point toron as representing the motion of free particles. The 'loops' of one or more torons which begin and end upon the branch point toron then represent modifications of the free particle propagators due to the fact that each particle actually is moving through a medium consisting of the other particles. In Fig. 10, we have roughly given a pictorial indication of this modification. The situation actually is more complicated than as pictured because, as is shown in Fig. 9, interaction lines may cross.

The next term in our expansion, P_2, is the sum of all graphs containing two branch points. Two of these are pictured in Figs. 9(e) and 9(f). As in the classical theory, we consider these graphs primarily to be the mathematical representation of two-body interactions, the potential of interaction being a screened potential. It is worth-while to expand upon this.

First, we remark that the 'loops' of Figs. 9(e) and 9(f), consisting of a chain of torons connecting a branch point to itself, represent the same modification of the free particle propagator as we discussed under P_1. Next, the chains of 0, 1, 2, . . . , torons connecting the two branch points represent interactions between particle pairs via a screened potential. This screened potential can in fact be explicitly evaluated by performing the $\mathbf{R}$ and β integrations for the vertices of the chain, and then summing over all length chains. The resulting potential, which is temperature dependent, is much more complicated than the screened potential of classical theory, and we have not analysed it as yet. However, one does find that its Fourier transform is finite at $k = 0$, so that it is of short range.

The fact that the theory we have constructed utilizes a screened potential (the physical basis of which is identical to that discussed in the preceding section) in treating two-body interactions is of considerable importance when one considers two-body bound states. If one treats such states for a normal Coulomb potential, a well-known difficulty, the ultraviolet catastrophe, arises. This difficulty, which is a divergence of the internal partition function for the two-body bound states, arises because of the long range of the Coulomb potential, since this long range causes a piling up of an infinite number of energy levels at zero energy. However, for a short range, screened, potential, such as is introduced as a *natural consequence* of our ordering process, the number of bound state energy levels is finite and the internal partition function is convergent. Stated differently, the integrand* for the second virial coefficient (or second cluster coefficient, since we are dealing with an activity expansion) is convergent instead of divergent at large $\mathbf{R}$.† We also remark at this point that the small $\mathbf{R}$ divergence, which occurs in the classical theory, should not appear in the quantum theory. This is an obvious consequence of the uncertainty principle, which does not allow an electron, e.g., to get too close to a proton.

Proceeding by analogy with the classical theory, succeeding terms in the expansion are defined by the statement that the n-th term, P_n, is the sum of all diagrams containing n branch points. Once more, P_n for $n \geqslant 3$, as for $n = 2$, represents the effect upon the pressure of n-body interactions, the interactions always taking place via screened potentials. We expect that this ordering of terms will, in the physical situation discussed in the first section, then lead to a convergent series expansion for the pressure of our plasma.

5. COMMENTARY

We conclude with a discussion of the unresolved aspects of this problem. First we mention again the troublesome diagrams containing torons of nodal order one. These we have eliminated by introducing a new condition of 'electrical neutrality,' and even then, as we mentioned, we are not quite sure that we have completely eliminated them. Clarification of this point is important, and failure to clarify it will

* The plural is more appropriate for the quantum calculation.
† Similar remarks apply to the higher coefficients.

necessarily cast some doubt upon the results of any calculation. It might be mentioned, however, that detailed calculation of the chain part of any diagram containing only one toron of nodal order one (hence only one chain) should be easy. Since only these diagrams can cause difficulty, it may prove desirable to calculate them and then make numerical estimates of their significance.

We next remark that our 'electrical neutrality' condition, equation (27), may not correspond to the normal condition, $\sum_i \rho_i C_i = 0$. We have in fact not yet been able to demonstrate their equivalence. If they were not equivalent, our calculation then would not correspond to an electrically neutral system—and the latter is presumably what we want! We point out, however, that it is hard to imagine thermal equilibrium in any non-neutral system, since one expects such a system to be stable only in the limit of zero density. In a calculation of the type we contemplate, any instability at finite density would presumably manifest itself in divergences in the terms we compute. It is just because we do not 'see' such divergences that we expect that equation (27) is equivalent to the normal condition (but perhaps we do not 'see' clearly enough).

Third, and most important, is the matter of the integrations. If these cannot be done, the classification scheme becomes pointless. We do anticipate some difficulty in this connexion, particularly for those diagrams for which the Faltung theorem cannot be employed because of the crossing of interaction lines (see Fig. (9)). However, it appears that the graphs comprising a given P_n can be ordered in terms of their topological complexity, and that this ordering corresponds to a low activity (hence also a low density) expansion of the given P_n. We can illustrate this idea for P_1. For simplicity we use Boltzmann statistics. For such statistics one can show that only torons of sectionality one occur (MONTROLL and WARD, 1958; LEVINE, 1960). The 'typical' diagram for P_1 may then be drawn as in Fig. 11. In this figure the long sausage (a distended circle) represents the single branch point. The looping railroad tracks, as indicated, represent the sum over all length loops of torons (minimum length = one toron). The diagram can be analysed in terms of the pieces shown in Fig. 11(c) to 11(f), each piece representing modified propagators of the type mentioned in the previous section. Each loop is of order one in the activity, since the order of a chain of p torons is p in activity and $p = 1$ for the shortest chain. Because of this the lowest order contribution to P_1 is given by summing those diagrams containing only the piece of Fig. 11(c), but not the pieces of Figs. 11(d), 11(e), (11f), etc. To next order, one must include Figs. 11(d) and 11(e), but not 11(f), etc. By utilizing decomposition schemes of this sort, one hopes to develop low-density, high-temperature expansions for P_1, P_2, etc.

Fourth, even if one succeeds in performing the calculations outlined above, one still has a pressure-

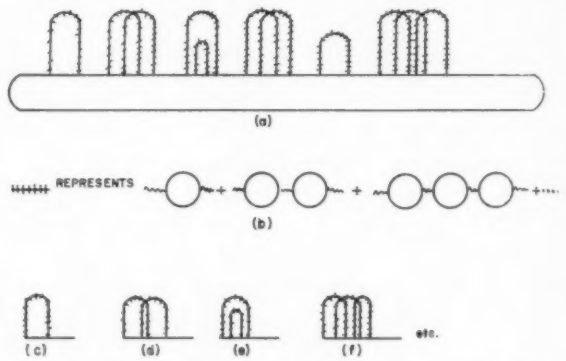


FIG. 11.—(a) Typical diagram for P_1 (Boltzmann statistics).
(b) Meaning of the 'railroad track.'
(c) Pieces which make up P_1.

temperature-activity relation. In order to get a pressure-temperature-density relationship one first would have to differentiate P with respect to the activities of the various components. In principle one then could invert the latter expressions to obtain the activities in terms of the densities and temperatures, and then by substitution in the original equation get a pressure-temperature-density equation of state. However, by comparison with equivalent classical calculations, we expect that such a process would be too complicated. One might instead directly use, for a system of Q components, the set of $Q + 1$ equations giving the pressure and the densities in terms of the temperature and the Q activities. One then would treat the activities as numerical parameters, and construct tables in which one looks up, at any temperature, the pressure for a given set of density parameters. We might conclude by remarking that this is certainly not the ideal situation. One really would like to start, as in the classical theory, from a virial rather than a cluster development, so as to directly obtain a pressure-temperature-density equation of state. Whether or not one can develop the necessary quantum generalization of MAYER'S (MAYER and MAYER, 1940) classical graph analysis of the virial series is for the future to decide.

Acknowledgement—I wish to thank DR. HUGH E. DEWITT for many stimulating conversations.

REFERENCES

COWAN R. D. and KIRKWOOD J. G. (1958) *J. Chem. Phys.* **29**, 264.

FRIEDMAN H. L. (1959) *Molecular Phys.* **2,** 23, 190.
GELL-MANN M. and BRUECKNER K. A. (1957) *Phys. Rev.* **106,** 364.
HARRIS G. M. Private communication.
LATTER R. (1955) *Phys. Rev.* **99,** 1854. (Several additional papers are listed in this and the Reference to COWAN and KIRKWOOD).
LEVINE H. B. (1960) *Phys. Fluids* **3,** 225.
MAYER J. E. (1950) *J. Chem. Phys.* **18,** 1426.
MAYER J. E. and MAYER M. G. (1940) *Statistical Mechanics,* Chap. 13. John Wiley, New York.
MEERON E. (1957) *J. Chem. Phys.* **26,** 804; (1958) *J. Chem. Phys.* **28,** 630.
MONTROLL E. W. and WARD J. C. (1958) *Phys. Fluids* **1,** 55. (For similar work, see also LEE T. D. and YANG C. N. (1959) *Phys. Rev.* 113, 1165).

VOL.
2
961

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 218 to 223. Pergamon Press Ltd. Printed in Northern Ireland

APPLICATION OF A MODIFIED DEBYE-HÜCKEL THEORY TO FULLY IONIZED GASES*

R. E. KIDDER and H. E. DEWITT
Lawrence Radiation Laboratory, University of California, Livermore, California

Abstract—The equations of the Debye-Hückel theory, modified to include quantum statistics, are discussed. It is found that the non-linear equations used by COWAN and KIRKWOOD are not unique, and that the non-linear theory can be formulated in different ways to give different answers. The linearized equations of these alternative formulations are discussed, and the correct form of the linearized theory is established. From the linear theory, the Helmholtz free energy of a slightly degenerate plasma is derived and, from this result, useful formulae in the near-classical limit are obtained for the pressure and internal energy.

1. INTRODUCTION

IN its linearized form, the classical Debye-Hückel theory of ionic solutions has been very useful in providing a quick derivation of the thermodynamic properties of a fully ionized gas in the classical limit ($\hbar = 0$). The results of this intuitive theory have been confirmed by a rigorous statistical-mechanical derivation using the MAYER cluster expansion of the canonical partition function (MONTROLL and MAYER, 1941). It has been shown that the linearized form of the Debye theory is equivalent to the summation of the ring integrals of the classical cluster expansion. The theory in its non-linear form, however, is beset with insurmountable difficulties when applied to a classical multicomponent gas of point-charged particles. From the success of the Thomas-Fermi statistical theory of the atom, it has been expected that by modifying the Debye-Hückel theory to the extent of describing the electrons with Fermi statistics instead of Boltzmann statisties the difficulties of the classical theory could be removed. Recently, KIRKWOOD and COWAN (1958) have made detailed numerical calculations of the thermodynamic functions of a dense plasma, using the non-linear equations of a Debye-Hückel-Thomas-Fermi theory. Also, LANDAU and LIFSHITZ (1958) have indicated how the linearized Debye theory may be ammended to include the effects of quantum statistics. Exact results for the thermodynamic functions of a slightly degenerate gas are now available (DEWITT, 1959). In this note we examine the KIRKWOOD-COWAN theory in its linearized form. It is found that the logic used by them enables one to write two alternative sets of equations which give different answers for the same problem. LANDAU and LIFSHITZ use a slightly different set of equations and obtain results that disagree with the exact theory because of the violation of a symmetry requirement. We take the point of view that the Debye-Hückel theory, when modified with quantum statistics should, in its linearized form, agree with the rigorous results obtained by summing the ring diagrams of the quantum-mechanical grand partition function. A form of the linearized theory that meets this requirement is given.

* This work was performed under the auspices of the U.S. Atomic Energy Commission.

2. THE BASIC APPROXIMATIONS AND THE EQUATIONS OF THE DEBYE-HÜCKEL–THOMAS-FERMI THEORY

We consider a fully ionized gas composed of electrorons with average density n_{-0} and ions of charge $z_i e$ and average density n_{+0}. Electrical neutrality requires that $z_i n_{+0}$ be equal to n_{-0}. Around each ion there is an average potential Ψ_+ due to the ion being a centre surrounded by a statistical distribution of positive and negative charge; similarly, around each electron an average potential Ψ_- is defined. These average potentials satisfy Poisson's equation:

$$\begin{aligned} \nabla^2\Psi_+ &= -4\pi z_i e\,\delta(r) - 4\pi\rho_+, \\ \nabla^2\Psi_- &= 4\pi e\,\delta(r) - 4\pi\rho_-. \end{aligned} \tag{1}$$

The δ functions express the boundary conditions, which are:

$$\begin{aligned} \lim_{r\to 0} r\Psi_+ &= z_i e, \\ \lim_{r\to 0} r\Psi_- &= -e. \end{aligned} \tag{2}$$

The average charge densities are given by:

$$\begin{aligned} \rho_+ &= e(z_i n_{++}(r) - n_{-+}(r)), \\ \rho_- &= e(z_i n_{+-}(r) - n_{--}(r)), \end{aligned} \tag{3}$$

where $n_{++}(r)$ and $n_{-+}(r)$ are the average particle-number densities of ions and electrons, respectively, at a distance r from a given ion. Similarly, n_{+-} and n_{--} are the densities of ions and electrons about a given electron. The average distribution of negative charge about an ion must be the same as the average distribution of positive charge about an electron; hence, we have the basic symmetry requirement:

$$n_{-+}(r) = z_i n_{+-}(r). \tag{4}$$

Since the electrons obey Fermi statistics, the average density of electrons about ions may be written as:

$$n_{-+}(r) = \left\langle \sum_{p_i} [1 + \exp(-\alpha_e + \beta p_i^2/2m + \beta w_{-+}(r, \alpha))]^{-1} \right\rangle_a, \tag{5}$$

where $w_{-+}(r, \alpha)$ denotes the energy of an electron at a distance r from an ion, in the presence of all the remaining ions and electrons in a particular configuration denoted by a. The averaging in (5) is taken over all configurations, with the distance r held fixed. An initial basic assumption of the Debye-Hückel approach is that equation (5) is approximated by:

$$n_{-+}(r) \cong \sum_{p_i} [1 + \exp(-\alpha_e + \beta p_i^2/2m + \beta \langle w_{-+}(r, \alpha)\rangle_a)]^{-1}. \tag{6}$$

This assumption is rigorously true only to first order in $w_{-+}(r, \alpha)$, since there is no reason to expect that $\langle w_{-+}{}^n \rangle$ is equal to $\langle w_{-+}\rangle_a{}^n$ for n greater than 1. Henceforth, the average electron-ion energy, $\langle w_{-+}(r,\alpha)\rangle_a$, will be denoted by $w_{-+}(r)$. If the summation of (6) over momentum states p_i is converted to an integration, the following equation, in the notation of the previous paper is obtained:

$$n_{-+}(r) = n_{-0}\zeta_e^{-1}\mathscr{J}_{1/2}(\alpha_e - \beta w_{-+}(r)), \tag{7}$$

where

$$\zeta_e = \frac{(2\pi\hbar)^3 n_{-0}}{(2s_e+1)\pi^{3/2}(2mkT)^{3/2}}$$
$$\mathscr{J}_{1/2}(\alpha) = \frac{4}{\sqrt{\pi}}\int_0^\infty \frac{dx\, x^2}{1+\exp(-\alpha + x^2)}. \tag{8}$$

Similarly, for the ions, we find:

$$n_{++}(r) = n_{+0}\zeta_i^{-1}\begin{cases}\mathscr{J}_{1/2}(\alpha_i - \beta w_{++}(r)) & \text{F.D.}\\ \mathscr{S}_{1/2}(\alpha_i - \beta w_{++}(r)) & \text{B.E.},\end{cases} \tag{9}$$

depending on whether the ions are fermions or bosons. The charge density must vanish as r approaches ∞, since the system is electrically neutral; this requirement determines the chemical potentials α_e and α_i in (7) and (9) from the equations:

$$\zeta_e = \mathscr{J}_{1/2}(\alpha_e)$$
$$\zeta_i = \mathscr{J}_{1/2}(\alpha_i) \quad \text{or} \quad \mathscr{S}_{1/2}(\alpha_i). \tag{10}$$

In the classical limit, $\hbar \to 0$, (7) and (9) reduce to Boltzmann statistics:

$$n_{-+}(r) = n_{-0}\exp(-\beta w_{-+}(r))$$
$$n_{++}(r) = n_{+0}\exp(-\beta w_{++}(r)). \tag{11}$$

Classically, the average energies go to $w_{-+}(r) \to -z_i e^2/r$ and $w_{++}(r) \to +z_i^2 e^2/r$ as r approaches 0. Consequently, $n_{++}(r)$ vanishes as r approaches 0, but $n_{-+}(r)$ has an essential singularity. This catastrophic divergence renders the classical non-linear Debye-Hückel equations inapplicable to a gas of point charges. The Fermi form (7) of $n_{-+}(r)$ is well-behaved at $r = 0$, since the repulsions due to the Pauli principle prevent the classical piling-up of negative charge around the ion.

By the same reasoning, the density of positive charge around an electron is given by:

$$n_{+-}(r) = n_{+0}\zeta_i^{-1}\begin{cases}\mathscr{J}_{1/2}(\alpha_i - \beta w_{+-}) & \text{F.D.}\\ \mathscr{S}_{1/2}(\alpha_i - \beta w_{+-}) & \text{B.E.},\end{cases} \tag{12}$$

and, similarly, for negative charge around the electron we have:

$$n_{--}(r) = n_{-0}\zeta_e^{-1}\mathscr{J}_{1/2}(\alpha_e - \beta w_{--}). \tag{13}$$

In order to obtain usable equations, it is next assumed that the various average energies are related to the average potentials as follows:

$$w_{++} = z_i e\Psi_+$$
$$w_{-+} = -e\Psi_+$$
$$w_{+-} = z_i e\Psi_-$$
$$w_{--} = -e\Psi_-. \tag{14}$$

The justification of this basic assumption and the other approximations of the Debye-Hückel theory are discussed in detail by FOWLER (1955).

KIRKWOOD and COWAN form a non-linear equation for Ψ_+ by using (7) for $n_{-+}(r)$ and the Boltzmann form for $n_{++}(r)$. The equation for Ψ_- is coupled to Ψ_+ by the use of the symmetry requirement (4). These equations are:

$$\nabla^2\Psi_+ = -4\pi z_i c\,\delta(r) - 4\pi e[z_i n_{+0}\zeta_i^{-1}\mathscr{J}_{1/2} \times (\alpha_i - \beta z_i e\Psi_+) - n_{-0}\zeta_e^{-1}\mathscr{J}_{1/2}(\alpha_e + \beta e\Psi_+)]$$
$$\nabla^2\Psi_- = 4\pi e\,\delta(r) - 4\pi e[n_{-0}\zeta_e^{-1}\mathscr{J}_{1/2}(\alpha_e + \beta e\Psi_+) - n_{-0}\zeta_e^{-1}\mathscr{J}_{1/2}(\alpha_e + \beta e\Psi_-)]. \tag{15}$$

(For generality, we have retained quantum statistics for the ions as well as electrons.) Using the same logic, however, one arrives at an alternative set of equations by first writing an equation for Ψ_-, and

then coupling the equation for Ψ'_+ to that for Ψ'_- with the symmetry requirement. This alternative set is:

$$\begin{aligned}\nabla^2\Psi'_- &= 4\pi e\,\delta(r) - 4\pi e[z_i n_{+0}\zeta_i^{-1}\mathscr{I}_{1/2} \\ &\quad\times(\alpha_i - \beta z_i e\Psi'_-) - n_{-0}\zeta_e^{-1}\mathscr{I}_{1/2}(\alpha_e + \beta e\Psi'_-)] \\ \nabla^2\Psi'_+ &= -4\pi z_i e\,\delta(r) - 4\pi e[z_i n_{+0}\zeta_i^{-1}\mathscr{I}_{1/2} \\ &\quad\times(\alpha_i - \beta z_i e\Psi'_+) - z_i n_{+0}\zeta_i^{-1}\mathscr{I}_{1/2}(\alpha_i - \beta z_i e\Psi'_-)].\end{aligned} \tag{16}$$

Both sets of equations, (15) and (16), reduce to the classical non-linear equations of the Debye-Hückel theory in the classical limit, $\hbar \to 0$. For positively and negatively charged particles of comparable mass, there is no reason to prefer one set over the other. However, for real electrons and nuclei with their tremendous mass difference, the set (15) used by KIRKWOOD and COWAN seems more reasonable for approximate numerical computation of the thermodynamic functions of a dense plasma.

To examine the consequences of the two alternative sets of equations, we look at the solutions of their linearized forms. The linearized equations are valid for r sufficiently large so that $\beta z_i e\Psi'_\pm$ and $\beta e\Psi'_\pm$ are much less than 1. The $\mathscr{I}_{1/2}$ (or $\mathscr{S}_{1/2}$) functions may be expanded in Taylor series to give:

$$\begin{aligned}\zeta_e^{-1}\mathscr{I}_{1/2}(\alpha_e + \beta e\Psi'_+) &= 1 + \beta e\theta_e^2\Psi'_+ + \ldots \\ \zeta_i^{-1}\mathscr{I}_{1/2}(\alpha_i - \beta z_i e\Psi'_+) &= 1 - \beta z_i e\theta_i^2\Psi'_+ + \ldots,\end{aligned} \tag{17}$$

where

$$\begin{aligned}\theta_e^2 &= \zeta_e^{-1}\mathscr{I}_{1/2}(\alpha_e) = \mathscr{I}_{-1/2}(\alpha_e)/\mathscr{I}_{1/2}(\alpha_e) \\ \theta_i^2 &= \zeta_i^{-1}\mathscr{I}_{1/2}(\alpha_i) = \mathscr{I}_{-1/2}(\alpha_i)/\mathscr{I}_{1/2}(\alpha_i).\end{aligned} \tag{18}$$

If (17) is used, the linearized equations obtained from (15) are:

$$A\begin{cases}\nabla^2\psi_+ = 4\pi\beta e^2[z_i^2\theta_i^2 n_{+0} + \theta_e^2 n_{-0}]\psi_+ \\ \nabla^2\psi_- = 4\pi\beta e^2\theta_e^2[z_i^2 n_{+0}\,\psi_+ - n_{-0}\,\psi_-],\end{cases} \tag{19}$$

and from (16):

$$B\begin{cases}\nabla^2\psi_- = 4\pi\beta e^2[z_i^2\theta_i^2 n_{+0} + \theta_e^2 n_{-0}]\psi_- \\ \nabla^2\psi_+ = 4\pi\beta e^2\theta_i^2[z_i^2 n_{+0}\psi_+ - n_{-0}\psi_-].\end{cases} \tag{20}$$

The solutions of the linearized equations are designated with ψ_+ and ψ_-, to distinguish them from Ψ'_+ and Ψ'_- for the non-linear equations, where ψ_+ and ψ_- are the asymptotic forms of Ψ'_+ and Ψ'_- for large r. The δ-functions expressing the boundary conditions at $r = 0$ have not been written in the linearized equations, since these boundary conditions are of no help in determining the integration constants in ψ_+ and ψ_-.

The general solutions of the linear equations A, subject only to the requirement that ψ_+ and ψ_- vanish at ∞, are:

$$\begin{aligned}\psi_+ &= \frac{z_i e}{r}\,C_{A+}e^{-\kappa r} \\ \psi_- &= -\frac{e}{r}\,[C_{A-}e^{-\kappa_-\theta_e r} + C_{A+}(\theta_e^2/\theta_i^2)e^{-\kappa r}],\end{aligned} \tag{21}$$

where

$$\begin{aligned}\kappa &= (\kappa_-^2\theta_e^2 + \kappa_+^2\theta_i^2)^{1/2} = 1/\lambda \\ \kappa_-^2 &= 4\pi e^2\beta n_{-0} \\ \kappa_+^2 &= 4\pi e^2\beta\, z_i^2 n_{+0}.\end{aligned} \tag{22}$$

Here, C_{A+} and C_{A-} are arbitrary constants, and κ is the reciprocal of the Debye screening length λ, generalized to include the modifications due to quantum statistics. Similarly, the general solutions of equations B are:

$$\begin{aligned}\psi_- &= -\frac{e}{r}\,C_{B-}e^{-\kappa r} \\ \psi_+ &= \frac{z_i e}{r}\,[C_{B+}e^{-\kappa_+\theta_i r} + C_{B-}(\theta_i^2/\theta_e^2)e^{-\kappa r}].\end{aligned} \tag{23}$$

By numerical integration of the non-linear equations (15) and comparison of ψ_+ and ψ_- with (21) for large r, we may find numerical values for C_{A+} and C_{A-}. Similarly, numerical values of C_{B+} and C_{B-} could be found by numerical integration of (16). The solutions, (21) and (23), of the alternative sets of linear equations are different from each other, and hence would lead to different answers for the thermodynamic functions of the gas. Furthermore, ψ_+ and ψ_- have the unsatisfactory feature of behaving differently as r approaches ∞. In (21), ψ_+ vanishes as $e^{-\kappa r}$, while ψ_- vanishes as $e^{-\kappa_-\theta_e r}$, since $\kappa_-\theta_e$ is less than κ.

The only way in which the solutions (21) and (23) can be identical is to have the condition: $C_{A-} = C_{B+} = 0$. If this requirement is made, then (20) and (23) have the same form and require only one arbitrary constant:

$$\begin{aligned}\psi_+ &= \frac{z_i e\theta_i^2}{r}\,Ce^{-\kappa r} \\ \psi_- &= -\frac{e\theta_e^2}{r}\,Ce^{-\kappa r}.\end{aligned} \tag{24}$$

In view of the non-uniqueness of the theory, as illustrated by two apparently equally valid sets of equations A and B, the solutions (24) are to be preferred over (21) or (23). In the classical limit, the degeneracy parameters θ_i^2 and θ_e^2 both reduce to 1. The remaining arbitrary constant C is determined by the fact that the Debye-Hückel theory is rigorously valid at sufficiently high temperature. The asymptotic forms

VOL. 2 1961

ψ_+ and ψ_- are valid arbitrarily close to the origin, and at the origin we must have $r\Psi_+$ equal to $z_i e$ and $r\Psi_-$ equal to $-e$; hence, C has the value 1 for a non-degenerate gas. It is clear, however, that for a degenerate gas (i.e. $\theta_e^2 \neq \theta_i^2 \neq 1$) the constant C cannot be chosen so that ψ_+ and ψ_- reduce to $z_i e/r$ and $-e/r$, respectively, as r approaches 0.

The chief criticism of the two sets of non-linear equations, (15) and (16), for the calculation of thermodynamic functions of a dense gas is that they are unsymmetrical in the way Ψ_+ and Ψ_- are treated. With the set of equations (14) used by KIRKWOOD and COWAN, Ψ_+ may be calculated independently of Ψ_-, but the calculation of Ψ_- requires a knowledge of Ψ_+. The same statement applies to the equations (16), but in reverse order. The two ways of doing the calculation, in general, will give different results. Thus, it would seem more reasonable to begin with a symmetrical set of equations in which Ψ_+ and Ψ_- are treated on an equal basis. The first equation of (15) and the first of (16) are symmetrical in this sense. An objection against this symmetrical set of equations for numerical calculations of the type done by KIRKWOOD and COWAN is that the symmetry requirement, $n_{-+}(r) = z_i n_{+-}(r)$, is violated for r small enough to cause non-linear effects to be important. We will show, however, that the symmetrical equations are suitable for developing a linearized Debye-Hückel theory which gives the correct results for the thermodynamic functions of a slightly degenerate gas.

The linearized symmetrical equations, namely, the first of A and the first of B, have the solutions:

$$\psi_+ = \frac{z_i e}{r} C_+ e^{-\kappa r}$$
$$\psi_- = -\frac{e}{r} C_- e^{-\kappa r}. \tag{25}$$

The symmetry requirement, $n_{-+}(r) = z_i n_{+-}(r)$, gives a relation between the two arbitrary constants C_+ and C_- in (25). The expansions of these charge densities are:

$$n_{-+}(r) = n_{-0}\{1 + \beta e\theta_e^2 \psi_+ + \cdots$$
$$z_i n_{+-}(r) = z_i n_{+0}\{1 - \beta e\theta_i^2 \psi_- + \cdots. \tag{26}$$

The two densities, n_{-+} and zn_{+-}, can be equal to $0(\psi)$ only if $\theta_e^2 C_+$ is equal to $\theta_i^2 C_-$, and (25) may then be written as (24). Thus, the equality of the solution of the unsymmetrical sets of linearized equations, A and B, is equivalent to preserving the charge-density symmetry in $0(\psi)$. The symmetry is violated in $0(\psi^2)$ for a degenerate gas and hence the usefulnesss of the non-linear symmetrical equations for numerical calculations is questionable.

LANDAU and LIFSHITZ, in their discussion of the degenerate plasma, used the symmetrical linearized equations and obtained the solutions (25). They used the boundary conditions at $r = 0$, however, and so put the constants C_+ and C_- equal to 1. This choice violates the symmetry requirement, even to first order in ψ.

3. CALCULATION OF THE THERMODYNAMIC FUNCTIONS

The average potential energy of the gas is given by:

$$\bar{U} = \tfrac{1}{2} N_i \int d\mathbf{r} \rho_+(z_i e/r) + \tfrac{1}{2} N_e \int d\mathbf{r} \rho_-(-e/r)$$
$$= \bar{U}_i + \bar{U}_e. \tag{27}$$

The solutions of the linearized equations (24) are valid for sufficiently large r, and with Poisson's equations, they give the asymptotic forms of the charge densities as:

$$\rho_{+\text{asy}} = -\frac{1}{4\pi} \nabla^2 \psi_+ = -\frac{z_i e\theta_i^2 \kappa^2}{r} Ce^{-\kappa r}$$
$$\rho_{-\text{asy}} = -\frac{1}{4\pi} \nabla^2 \psi_- = \frac{e\theta_e^2 \kappa^2}{r} Ce^{-\kappa r}. \tag{28}$$

The asymptotic forms (28) are accurate for distances from the central point charges greater than some limit to be denoted as r_e. Inside the sphere of radius r_e, the average potential and charge density are not known, though presumably they may be calculated approximately from the non-linear equations, either (15) or (16). The average energy integral is thus a sum of two parts:

$$\bar{U}_i = \tfrac{1}{2} N_i 4\pi \int_0^{r_e} dr r^2 \rho_{+\text{exact}}(z_i e/r) + \tfrac{1}{2} N_i 4\pi \times \int_{r_e}^{\infty} dr r^2 \rho_{+\text{asy}}(z_i e/r). \tag{29}$$

The quantity $\bar{U}_i$ may be estimated by using $\rho_{+\text{asy}}$ for the entire volume, even for r less than r_e where it is not accurate, since the main contribution comes from r greater than r_e. With this approximation and using (28), we find:

$$\bar{U}_i \simeq \tfrac{1}{2} N_i 4\pi \int_0^{\infty} dr r^2 \rho_{+\text{asy}}(z_i e/r) = -\tfrac{1}{2} C N_i z_i^2 e^2 \theta_i^2 \kappa. \tag{30}$$

Adding a similar result for $\bar{U}_e$, we have, for the potential energy of the gas due to the particle correlation,

$$\bar{U}/V = -\tfrac{1}{2} Ce^2 (z_i^2 \theta_i^2 n_{+0} + \theta_e^2 n_{-0})\kappa$$
$$= -\frac{CkT\kappa^3}{8\pi}. \tag{31}$$

We have seen that the remaining arbitrary constant C must be 1 in the classical limit, but for a degenerate

gas, we cannot rule out the possibility that C is a function of temperature in such a way that lim $C(T)$ is equal to 1 as T approaches ∞. Equation (31), however, agrees with the exact near-classical-limit result for $\bar{U}$ as derived from the grand partition function (equation (122) of the previous paper) when C is equal to 1.

The Helmholtz free energy may be obtained from $\bar{U}$ by an integration with respect to the coupling constant from 0 to e^2; this calculation is equivalent to the Debye charging process. Noting that $\bar{U}$ is $0(e^3)$, we easily find the contribution to the Helmholtz free energy to be:

$$F_e = \int_0^e \frac{d(e'^2)}{(e'^2)}\, \bar{U}_{e'} = -\frac{VkT\kappa^3}{12\pi}, \tag{32}$$

or, in the notation of the previous paper,

$$\frac{kT}{F_e} = -\frac{2}{3} N\Lambda\phi^{3/2}, \tag{33}$$

where

$$\begin{aligned} &N = N_e + N_i, \quad n = n_{+0} + n_{-0}, \quad f_e = N_e/N \\ &\Lambda = \sqrt{\pi} e^3 \beta^{3/2} n^{1/2}, \quad \phi = z_i^2 \theta_i^2 f_i + \theta_e^2 f_e. \end{aligned} \tag{34}$$

If the free energy due to the interaction is added to F_0, the free energy of an ideal quantum gas which is given by:

$$\frac{F_0}{kT} = N\{\alpha - \mathscr{J}_{3/2}(\alpha)/\mathscr{J}_{1/2}(\alpha)\}, \tag{35}$$

then the total free energy is given by:

$$F = F_{oi} + F_{oe} + F_e.$$

The pressure and internal energy are obtained from the expresssions:

$$\begin{aligned} \frac{PV}{kT} &= -V\frac{\partial}{\partial V}\left(\frac{F}{kT}\right) \\ \frac{E}{kT} &= \beta\frac{\partial}{\partial\beta}\left(\frac{F}{kT}\right). \end{aligned} \tag{36}$$

The differentiations of (36) are easily carried out by noting that from the defining relation for the chemical potential (10) we obtain:

$$\begin{aligned} V\,\partial\alpha/\partial V &= 1/\theta^2 \\ \beta\,\partial\alpha/\partial\beta &= 3/2\theta^2. \end{aligned} \tag{37}$$

Also, we need the relation:

$$\begin{aligned} V\frac{\partial}{\partial V}\left(\frac{\mathscr{J}_{-1/2}(\alpha)}{\mathscr{J}_{1/2}(\alpha)}\right) &= \left[\frac{\mathscr{J}_{-3/2}(\alpha)}{\mathscr{J}_{1/2}(\alpha)} - \frac{\mathscr{J}^2_{-1/2}(\alpha)}{\mathscr{J}^2_{1/2}(\alpha)}\right] V\frac{\partial\alpha}{\partial V} \\ &= -\left(\frac{\mathscr{J}_{-1/2}(\alpha)}{\mathscr{J}_{1/2}(\alpha)} - \frac{\mathscr{J}_{-3/2}(\alpha)}{\mathscr{J}_{-1/2}(\alpha)}\right) \equiv -\eta(\alpha). \end{aligned} \tag{38}$$

From (36), using (37) and (38), we obtain the following expressions for the pressure and energy:

$$\begin{aligned} \frac{PV}{kT} &= \Sigma N_i \frac{\mathscr{J}_{3/2}(\alpha_i)}{\mathscr{J}_{1/2}(\alpha_i)} - \frac{1}{3} N\Lambda\phi^{3/2} + N\Lambda\phi^{1/2}\Sigma z_i^2 f_i \eta(\alpha_i) \\ \frac{E}{kT} &= \frac{3}{2}\Sigma N_i \frac{\mathscr{J}_{3/2}(\alpha_i)}{\mathscr{J}_{1/2}(\alpha_i)} - N\Lambda\phi^{3/2} + \frac{3}{2} \\ &\quad \times N\Lambda\phi^{1/2}\Sigma z_i^2 f_i \eta(\alpha_i), \end{aligned} \tag{39}$$

where the sums are over the particle species of a multicomponent gas. Expressions (39) are identical with (127) and (128) of the previous article, except for the absence of the exchange interaction term.

4. CONCLUSION

The pressure and internal energy formulae (39) are exact in the near classical limit defined by high temperature and low density such that $\Lambda\phi^{3/2} = (8\pi\rho\lambda^3)^{-1} \ll 1$, *but* $\hbar \neq 0$. In this limit, the ring diagrams of the expansion of the quantum-mechanical grand partition function give the leading contribution to PV/kT after the ideal gas term (namely, $0(\Lambda)$), and the next contribution from non-ring terms is $0(\Lambda^2)$. We have thus shown that the linearized Debye-Hückel-Thomas-Fermi theory is exact in the near-classical limit in the same way in which the Debye-Hückel theory is exact in the classical limit ($\hbar = 0$). To distinguish between the various ways of building up such a theory, we must require that the Poisson equations be symmetrical in the treatment of the average potentials, and preserve the charge-density symmetry, $n_{-+} = z_i n_{+-}$ to $0(\psi)$. The usefulness of the Debye-Hückel method has thus been extended to more dense plasmas, but at the same time the limitations of the method are more apparent. The linearized theory gives the thermodynamic functions of essentially *classical* particles whose velocity distributions are slightly modified from the Boltzmann form, due to quantum statistics. Exchange interactions and wave-mechancial effects are in no way included in either the linear or the non-linear equations. Furthermore, it is not clear that the non-linear equations give any more information than is contained in the linearized forms. The non-linear effects of the symmetrical non-linear equations (the final equation of (15) and the first of (16) are not believable, since the charge symmetry, $n_{-+} = z_i n_{+-}$, is violated in $0(\psi^2)$. The set of non-linear equations (15) used by COWAN and KIRKWOOD are better in that the charge symmetry is preserved, but the set (16), which gives different answers, would seem to be equally valid. Until more exact results have been obtained from the non-ring

VOL. 2 196

terms of the partition function, and can be used for comparison, we conclude that the Debye-Hückel-Thomas-Fermi theory, as applied to plasmas, is useful only in its linearized form.

REFERENCES

COWAN R. D. and KIRKWOOD J. G. (1958) *J. Chem. Phys.* **29**, 264

FOWLER R. H. (1955) *Statistical Mechanics* (2nd Ed.) pp. 269-74, Cambridge University Press.

LANDAU L. D. and LIFSHITZ E. M. (1958) *Statistical Physics*, pp. 229-233, Pergamon Press, London.

MONTROLL E. W. and MAYER J. E. (1941) *J. Chem. Phys.* **9**, 626.

WITT H. E. DE (1959) *Thermodynamic Functions of a Partially-degenerate, Fully Ionized Gas*, University of California Radiation Laboratory Report No. UCRL-5652.

OL.
2
961

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 224 to 234. Pergamon Press Ltd. Printed in Northern Ireland

EQUILIBRIUM PROPERTIES OF A PARTIALLY IONIZED PLASMA*

G. M. HARRIS and J. TRULIO
Lawrence Radiation Laboratory, University of California, Livermore, California

Abstract—A model for a partially-ionized, partially-dissociated plasma has been formulated using known theoretical concepts to describe both bound and free electron states, internal molecular degrees of freedom and Coulomb interactions. It has been applied to a system of particles arising from the hydrogen molecule. The Coulomb interaction is treated in the classical Debye approximation. However, a distance of closest approach between ions and electrons depending on the kinetic energy of the electrons is included to avoid the short range divergence of the Coulomb potential. The kinetic energy of the free electrons is calculated from the partition function for a perfect Fermi gas. The vibrational and rotational motion is treated in the harmonic oscillator and rigid rotor approximation with the number of energy levels counted for a given electronic state depending on the dissociation energy of the state. A volume dependence of the bound electronic energy eigenvalues is included by considering the effect of surrounding particles as a confinement of a given particle to a spherical box of variable size. For the counting of the bound electronic states, a given state is bound until its energy increases to zero due to confinement. From the partition function for the entire system, the free energy is calculated. By a minimization of the free energy of the system, the equilibrium composition as a function of temperature and volume is obtained. Then not only can thermodynamic quantities be calculated, but it is believed that a reasonable approximation to the correct balance of molecular, ionic and free electonic states is achieved over a wide range of V–T space. Consequently, regions where incomplete ionization and dissociation are important are delineated. In addition, for different regions of V–T space, the relative contributions of charged particle interactions, non-classical behaviour of electrons, internal degrees of freedom and translation to the total energy of the system can be determined.

1. INTRODUCTION

CONSIDERABLE progress has been made in using diagram methods to derive the equilibrium properties of systems of charged particles (MAYER, 1937; MONTROLL and MAYER, 1941; MEERON, 1947, 1958; FRIEDMAN, 1959a, 1959b; LEE and YANG, 1959 MONTROLL and WARD, 1958; GLASSGOLD, HECKROTTE and WATSON, 1959; LEVINE, to be published; DEWITT, to be published). This approach is of particular interest theoretically for its rigour and generality. In principle one proceeds from fundamental laws to exact expressions for the macroscopic properties of arbitrary systems. These expressions ideally take the form of rapidly converging infinite series whose terms are tractable definite integrals. However, this ideal has not yet been attained even for a two-component plasma of electrons and protons. To calculate the properties of multicomponent systems, we have therefore adopted another procedure, a 'model' approach, involving many *ad hoc* assumptions based on physical intuition.

In this model a multicomponent system is a collection of subsystems which are molecules, atoms, ions and free electrons. The interactions of electrons and nuclei within these subsystems are taken into account in their internal electronic, vibrational and rotational partition functions. Electronic eigenvalues are made functions of the volume per subsystem. The functional dependence is based on that of the eigenvalues of a hydrogen atom in a spherical box (DEGROOT and TEN SELDAM, 1946). The free electrons are treated as a perfect Fermi gas. Electrostatic interactions among charged particles are calculated in the Debye (FOWLER and GUGGENHEIM, 1952) approximation, modified to distinguish between bound and free electrons. For this purpose, ions are regarded as single particles with an effective net charge; the free electron cloud around any positive charge centre has zero density within a sphere whose radius depends on the average kinetic energy of the free electrons.

Interactions within the subsystems contribute terms to the partition function which are analogous to classes of localized interactions in the diagram formalism which have not yet been evaluated. Their inclusion in this model extends the applicability of the calculation to the dilute, low temperature region. In addition, the inclusion of pressure ionization, i.e. the disappearance of bound states with increasing density, assures reasonable behaviour in regions of intermediate densities and temperatures. Thus this method allows the calculation of equilibrium properties over a much wider range than the more rigorous approaches now being studied.

With this model there is a direct indication of the

* This work was performed under the auspices of the U.S. Atomic Energy Commission.

relative importance of various interactions in different density-temperature regions. For example, regions where the concentration of multiparticle subsystems is important can be determined. Also the relative contributions to the free energy of Coulomb interactions, electron degeneracy, and internal degrees of freedom can be ascertained. Such information is very helpful in deciding which aspects of the model need most to be improved. Thus an analysis of the results from a set of assumed interactions serves as a guide to an improved model (HARRIS, 1959). It is important that new results can be obtained from an improved model with relatively little additional effort. It thereby becomes feasible to test one's intuitive ideas of the relative importance of various physical effects and particle groupings. Ultimately, the quantitative utility of this method of calculating the properties of multicomponent systems will rest on agreement with experiment.

In the next section a more detailed description is presented and a brief general discussion of the model which is currently incorporated in a working computer program. It is followed by a presentation and discussion of some results obtained when the method is applied to a system of particles arising from the hydrogen molecule.

2. MODEL AND METHOD OF CALCULATION

The system considered is an equilibrium distribution of atomic, ionic and molecular species arising from one mole of original molecular substance. The *ad hoc* assumptions of the model are all contained in the approximations used to calculate the free energy of the system. The free energy is calculated as the sum of contributions from the translation of the classical nuclear subsystems, from the translational motion of perfect Fermi electrons, from the internal degrees of freedom of the molecules and from a modified classical electrostatic interaction between charged particles. Interactions between neutral particles are neglected and we consider only small molecules. The calculated Helmholtz free energy is minimized with respect to the concentration of each independent component. The equilibrium distribution of species so obtained is then used to calculate all thermodynamic quantities of interest.

A. *Model—The free energy contributions*

(1) *Translation of perfect classical particles.* This free energy, F_1, is given exactly by

$$F_1 = kT\sum_i N_i \ln N_i - kT(3/2 \ln kT + 6{\cdot}974 + \ln MW + \ln v)\sum_i N_i - 3kT/2 \sum_i N_i \ln M_i$$

where

N_i = number of molecules of species i per molecule

M_i = atomic weight of ith species

v = volume (cm^3/g)

kT is in electron volts

MW = molecular weight of original molecular species.

(2) *The electronic, vibrational and rotational partition function.* Experimental data, referred to a common zero of energy, were used to obtain energy eigenvalues for the bound electronic states of each component. For the H atom, the electronic energies were made functions of the volume by assuming each atom to be in a spherical box of a size equal to the average volume per particle in the system (DEGROOT and TEN SELDAM, 1946). For more complicated components, approximate expressions were formulated, based on the results for the hydrogen atom. Appendix 1 describes the details of the analytic expression used for each species. When the size of the box decreased to the point where the energy of a given confined bound state became zero, that state was no longer counted. Hence, in addition to the energy eigenvalues, the number of bound electronic states for each component is also a function of the volume, and is finite for any finite volume. The harmonic oscillator and rigid rotor approximations were used for the vibrational and rotational partition functions. The number of these states was determined from the dissociation energy of the given electronic state. Thus the total internal partition function for a given component is given by:

$$Q_i = \sum_{k=1}^{k_{\max}(v)} \sum_{l=0}^{l_{\max}(ki)} \sum_{j=0}^{j_{\max}(l)} (2j+1) g_{ki} g_{ki} \times (R) e^{-[E_{ki}(v) + W_{ki} l + B_{ki} \cdot j \cdot (j+1)]/kT}$$

where

$E_{ki}(v)$ = energy of kth eigenstate of species i at volume v

g_{ki} = degeneracy of kth electronic state

W_{ki} = vibrational frequency of kth electronic state

l = vibrational quantum number

B_{ki} = rotational constant of kth electronic state

j = rotational quantum number

$g_{ki}(R)$ = fractional weighting factor used to avoid discontinuity in Q as a function of volume; $g_{ki}(R) \neq 1$ only for the last state counted which is the one of highest negative energy.

The cut-off of the vibrational and rotational states is determined by the relationship:

$$Do_{ki} = W_{ki}l + B_{ki} \cdot j \cdot (j+1)$$

with

Do_{ki} = dissociation energy of kth electronic state of ith species.

$$l_{\max}(j=0) = Do_{ki}/W_{ki}$$

and a $j_{\max}$ is determined for each l. Then the total internal free energy

$$F_2 = -kT \sum_i N_i \ln Q_i.$$

(3) *The perfect quantum electron gas.* The free energy of a perfect Fermi gas is given by

$$F_3 = N_e kT \left[\nu - \frac{2/3 I_{3/2}(\nu)}{I_{1/2}(\nu)} \right]$$

where ν = the chemical potential g/kT and

$$I_n(\nu) = \int_0^\infty x^n \, dx/[1 + \exp(-\nu + x)].$$

The $I_{1/2}$ integrals are related to the density of electrons by the normalization condition:

$$I_{1/2}(\nu) = \frac{N_e}{V(kT)^{3/2}} \frac{h^3}{4\pi(2m_e)^{3/2}}$$

where N_e = number of electrons.

Through this last equation (ν) is implicitly given in terms of the specified variables (N_e, V, T). Hence given (N_e, V, T), we can find $I_{1/2}(\nu)$, ν, and $I_{3/2}(\nu)$. For this purpose we have used a four-term asymptotic expansion for $\nu > 4$, tables (McDOUGALL and STONER, 1938) relating these quantities for $4 > \nu > -4$, and the classical expression for $\nu < -4$. We thereby calculate F_3 to within 0·3 per cent for all values of (N_e, V, T).

(4) *The electrostatic free energy* (F_4). According to the Debye-Hückel model, the contribution to the free energy due to electrostatic interactions is:

$$F_4 = -2\pi^{1/2}e^3 \sum_i (N_i z_i^2)^{3/2} \tau(\kappa a)/3(VkT)^{1/2}$$

where N_i = number of particles of charge z_i

a = distance of closest approach of the centres of 2 ions

$$J(\kappa a) = 3(\ln(1 + \kappa a) - \kappa a + 1/2(\kappa a)^2/(\kappa a)^3)$$

$$\kappa^2 = 4\pi e^2 \Sigma N_i z_i / VkT.$$

This expression includes a contribution from the interaction of positive ions with all electrons acting as classical charged particles. As a result, if used with arbitrary a it would allow electronic states with a continuum of negative energies decreasing without bound to minus infinity in the limit of a equal to zero. However, the free-energy contribution from the bound state interactions of electrons and nuclei has already been included quantum mechanically and is in fact given by F_2. Therefore we want to treat the interactions of free charged particles only by the Debye-Hückel method. Free electrons are defined as those of total positive energy. The total energy of a free electron is calculated assuming that all free electrons have the same kinetic energy $\bar{E} = kT(I_{3/2}(\nu)/I_{1/2}(\nu))$ which is the average energy per particle in a perfect Fermi gas. It is also assumed that the potential energy of an electron about a positive charge centre is due only to its interaction with that centre, although to be more consistent; the Debye-Hückel potential ought to be used. Then the potential energy of an electron at a distance r_i from a centre of charge z_i is $V(r_i) = z_i e^2/r_i$. The quantity a in the Debye expression is interpreted as the distance between the centres of an electron and a stationary ion for which the total energy of the electron is zero, i.e. $\bar{E} + V(a_i) = 0$. Thus we obtain $a_i = z_i e^2/\bar{E}$ giving a dependence of the potential energy of the electrons on their kinetic energy through the variable distance $a_i(T, v)$.

We have applied the electron-positive ion cut-off radius (a) to all pairwise interactions. However, the distance of closest approach of like particles should be zero, since the total energy is always positive for repulsive Coulomb interactions. By thus underestimating the repulsive interactions, we obtain a larger net attractive Coulomb energy than we should. To be completely consistent in using this model, there ought to be a set of cut-off diameters (a_{ij}) one for each type of ion (j) with a central ion of type (i). This modification together with the use of the Debye potential energy will be made in later calculations.

(5) *Consistency of the free energy formulation.* In this model two different kinds of distances have been defined which should be self-consistent. They are the average radius per particle, R, used to calculate the energy of the bound electronic state in a spherical box and the ionic radius a used in the electrostatic free energy F_4. The condition $a < R$ means that the system is dilute enough for the free electron to be the required distance away from any one particle and still be within the particle volume. However, for $a > R$ an electron cannot be outside the excluded volume of any one particle without being within that volume for another. The quantity a has meaning only for positive ion–electron interactions. Defining R' to be an average radius per positive ion, in a volume-temperature region where $a/R' > 1$, the assignment of bound electrons to one or the other of a few small subsystems becomes a questionable approximation. In such a region larger interacting clusters must be considered.

VOL 2 196

By calculating the quantity a/R' as well as $I_{1/2}$, and the four contributions to the free energy at every point in a specified temperature-volume grid, the general area of applicability of this model can be determined.

B. *Method of calculation*

At each temperature and volume the free energy is minimized with respect to the relative concentrations of all the independent species assumed to be present in the system. The conditions of charge neutrality and conservation of each kind of nucleus reduces the number of independent particle concentrations to $X = N - (A + 1)$, where N is the total number of species assumed present and A is the number of different nuclei. In the electronic partition function, the energies and the number of states are dependent on the average volume per particle. Also, in the electrostatic free energy the ionic radii are volume dependent through the dependence of the kinetic energy of a Fermi gas on particle density. Thus, the free energy has a complicated concentration dependence. For this reason the minimum value of F is found most conveniently by a direct search in a finite grid of the independent concentration variables. A 704 code was developed to do this for specified temperature-volume points. A systematic search over each grid size is made by varying 1 to X concentrations simultaneously in turn. Then, by repeatedly decreasing the grid size, a minimum value of F with an accuracy of 1 part in 10^5 is obtained together with the composition corresponding to this minimum. This then is the equilibrium composition of the system.

Once the free energy and equilibrium composition of the system is obtained for a given temperature and volume, the pressure and entropy of the system are calculated to a precision of 0·1 per cent in the same code by a three-point numerical differentiation from the relationships:

$$P = -\left(\frac{\partial F}{\partial V}\right)_{N_i, T} \quad \text{and} \quad S = -\left(\frac{\partial F}{\partial T}\right)_{N_i, V}$$

where the N_i are the equilibrium concentrations of the N species present. Then the internal energy $E = F + TS$, the Gibbs' free energy $A = F + PV$ and the enthalpy $H = E + PV$ are all directly calculated.

C. *Specific application of model*

The model just described has been applied to the calculation of the properties of a multicomponent system arising from a hydrogen molecule. Six species were assumed to be present: H_2, H_2^+, H, H^-, H^+ and electrons. In the calculation of F_2 the energies of the two stable states of both H_2^+ and H^- were used. Of the infinite set of bound states possible for an isolated H atom, only 37 states were still bound in the maximum size spherical box considered in this calculation, i.e. a specific volume of 4×10^{-30} cm^3/molecule. For H_2 we included the two Rydberg series of singly excited states which terminate in the two stable states of H_2^+. In addition all doubly excited states with energies between the ground states of these two series were included.

As noted previously, the energy of a hydrogen-like orbital is known exactly as a function of the volume of a confining sphere. For the other species a more approximate relationship between bound state energies and volume was used. The electronic energy of each state was represented as a sum of the energies of one-electron hydrogen like orbitals with different effective nuclear charges. These were chosen to reproduce the excitation and ionization energies of the isolated species when these were known. In the remaining instances, the energies of the isolated molecules were estimated by interpolation. For many excited states, molecular constants used in the rotational and vibrational energies were also estimated by interpolation.

For the electrostatic free energy, F_4, the ions, protons and electrons were all treated as single particle charges with the same pairwise distance of closest approach. The total free energy was minimized in a grid of four independent concentration variables for a given set of temperature-volume points. The calculations cover the temperature range $0{\cdot}025 \leqslant T \leqslant 100$ eV; and the volume range $0{\cdot}1 \leqslant V \leqslant 12{,}000$ cm^3/g.

3. RESULTS

A. *Regions of importance of the partial free energies*

Fig. 1 is a schematic representation of the behaviour of the H_2 system over the entire V–T region considered. Table 1 explains and defines the meaning of the Roman numeral designations of the figure. To define the regions of Fig. 1, we arbitrarily label free energy contributions of less than 1 per cent as negligible and greater than 10 per cent as strong. If a quantity other than the free energy had been used as a basis for comparison, somewhat different definitions would apply. It should be noted that there is no region of a perfect quantum gas and in all regions except the one that is called metallic, Coulomb interactions are more important than electron degeneracy.

B. *Composition variation*

Fig. 2 illustrates the composition variation with volume and temperature. The most striking result of this model is the effect of particle density on the degree

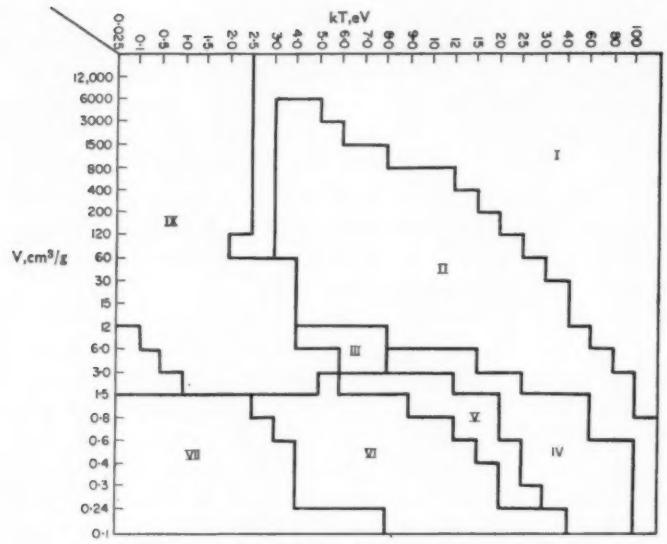


FIG. 1.—Schematic representation of the behaviour of H_2 system.
Note that $V = 12$ cm³/g corresponds to the normal solid density of H_2.

of ionization. The system is completely ionized for all temperatures at infinite volume. If the bound state electronic energies were independent of volume, then the number of free electrons would decrease monotonically as a function of density. This would occur since the free electron translational energy increases and the number of states per energy interval decreases as the density increases. However, with this model, the bound state energies also increase with density and the number of bound states decreases due to the confinement distortion of the bound state electronic wave functions. Thus at some density, free electron states can once again be favoured relative to bound states and the per cent ionization can increase with density. We do indeed find this behaviour characteristic of 'pressure ionization.' The percentage ionization increases from infinite volume to a minimum at some finite density where it then increases to complete ionization at large density. The present model for the volume dependence of the bound electronic eigenvalues, predicts that there can be no bound states at any temperature for volumes smaller than fifteen-fold liquid density (0·8 cm³/g). At higher densities, we have a totally ionized plasma. For low temperatures, i.e. $T \leqslant 7$ eV, total ionization occurs as a nearly discontinuous increase in the degree of ionization with increasing density. This is most extreme for $T = 1$ eV where we have nothing but H_2 molecules and H atoms up to almost fifteen-fold liquid density and complete ionization at slightly greater density. For the range $T = 1{\cdot}5$–5 eV there is a gradual decrease in ionization to zero with a nearly discontinuous rise from zero to complete ionization. Af $T = 6$ and 7, there is a gradual decrease in ionization but the minimum percentage ionization is not zero. The rise from the minimum to complete ionization is still almost discontinuous. Finally at $T \geqslant 8$ eV, there is a smooth decrease in ionization to a minimum with a smooth increase to complete ionization. The higher the temperature the larger the minimum amount of ionization and the lower the volume at which complete ionization occurs. Finally at $T \geqslant 50$ eV there is complete ionization for

TABLE 1.—INTERACTION REGIONS OF FIG. 1

Region	Description	Definition
I	Perfect classical gas of point particles	$\lvert\Delta F_2\rvert$,* $\lvert\Delta F_3\rvert$,** $\lvert F_4\rvert < 0{\cdot}01\,\lvert F_1\rvert$
II	Classical gas of interacting point particles	$10 > \lvert F_4\rvert > 1$;† $\lvert F_2\rvert$, $\lvert\Delta F_3\rvert$ negligible‡
III	Classical interacting gas with internal degrees of freedom	$10 > \lvert F_4\rvert, \lvert\Delta F_2\rvert > 1$; ΔF_3 negligible
IV	Quantum gas of interacting point charges	$10 > \lvert F_4\rvert, \lvert\Delta F_3\rvert > 1$; ΔF_2 negligible
V	Quantum gas of strongly interacting point charges	$\lvert F_4\rvert > 10$; $10 > \lvert\Delta F_3\rvert > 1$; ΔF_2 negligible
VI	Highly degenerate gas of strongly interacting point charges	$\lvert F_4\rvert, \lvert\Delta F_3\rvert > 10$; ΔF_2 negligible
VII	Metallic. Comparable potential and kinetic free energy	$\lvert F_2\rvert \simeq \lvert F_4\rvert$; F_1, ΔF_2 negligible
VIII	Molecular solid	$\lvert\Delta F_2\rvert$ largest; F_1, F_4, ΔF_3 negligible
IX	Perfect classical gas with internal degrees of freedom	$10 < \lvert\Delta F_2\rvert > 1$; F_4, ΔF_3 negligible

* Included in F_2 is a factor 15·833 N_1 which is just a correction for the zero of energy from isolated protons and electrons to the ground state of the H_2 molecule. In calculating the contribution of the electronic vibrational and rotational states to the free energy, this factor is subtracted.

** F_2 is the total kinetic free energy of the electrons. To determine the effect of the quantum statistics only the difference in the value of F_2 for classical electrons and the calculated value is significant, i.e. $\Delta F_3 = F_2 - F_2$ (classical). It is the magnitude of ΔF_3 relative to F_1 which is used to determine the degree of degeneracy of the electrons. The spin degeneracy of the electron remains in the classical limit and is hence not included in the quantum effect.

† The range of value of the F's are given in per cent relative to $\lvert F_1\rvert$ the classical kinetic free energy.

‡ Negligible means <1 per cent of $\lvert F_1\rvert$.

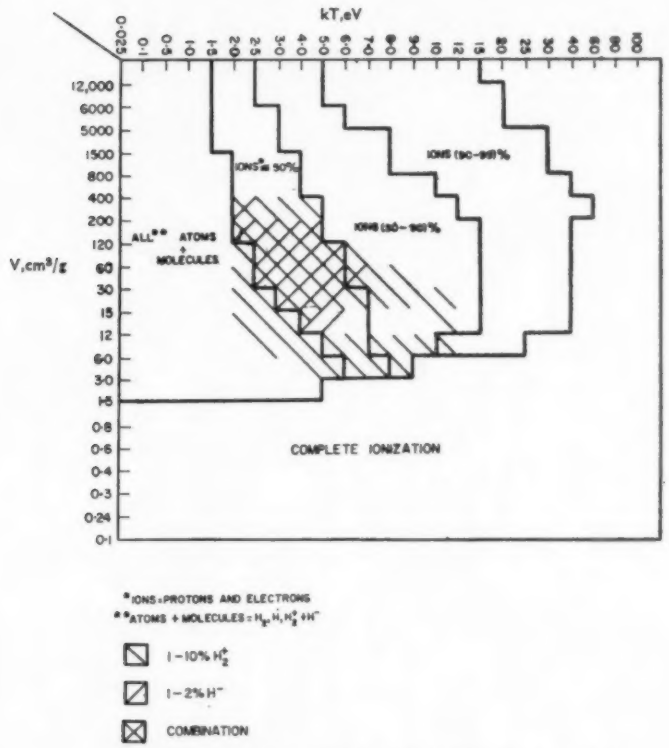


FIG. 2.—Schematic representation of composition variation.

all densities studied. Fig. 3 illustrates the effect of density on the degree of ionization in these various temperature regions. It is believed that the discontinuity which occurs at the lower temperatures is the result of the omission of larger clusters of subsystems. With the present model an electron is either free in the entire system or it is bound in a 2, 3 or 4 particle subsystem. Forced to maintain one or the other of these rather extreme distributions, the system as a whole has no way of freeing electrons in a continuous manner. The presence of psuedo-particles of the type: H_n^{+s} with $n \geqslant 3$ and $n > z > 0$ which free one electron per fraction of nucleus and in which bound electrons are not severely localized, would allow a more gradual transition from molecules to ion. At higher temperatures, due to ionization and dissociation, the presence of such large systems would be negligible. This fact accounts for the smooth variation of the number of free electrons with density for $T > 8$ obtained with the present system.

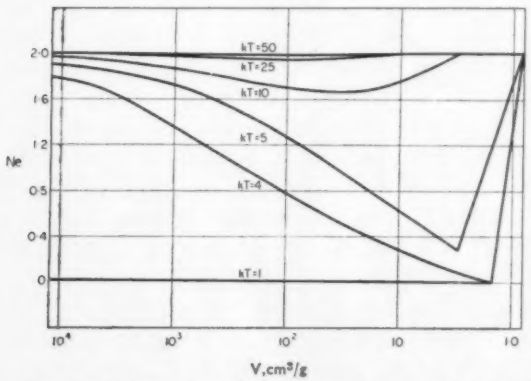


FIG. 3.—Volume dependence of the degree of ionization. $Ne = 2$ corresponds to 100 per cent ionization.

The degree of ionization as a function of temperature is also affected by the description of the bound states. The number and energy of the bound states depends not only on total volume, but also, through particle density, on temperature. Since the number of particles increases with temperature, the energy of the bound states increases and the number of bound states decreases as the temperature is increased at constant volume. Thus the ordinary temperature effect on ionization is enhanced. This takes place continuously for compressions below four-fold liquid density, where the degree of ionization is a smoothly increasing function of the temperature. At two-fold liquid density the last of the bound states for H^- disappears but since this is always present in trivial amounts, it has little effect on the rate of ionization. At four-fold liquid density, the ground state of the H atom disappears at $T = 8$ and of H_2 and H_2^+ at $T = 9$, giving complete ionization. This leads to a

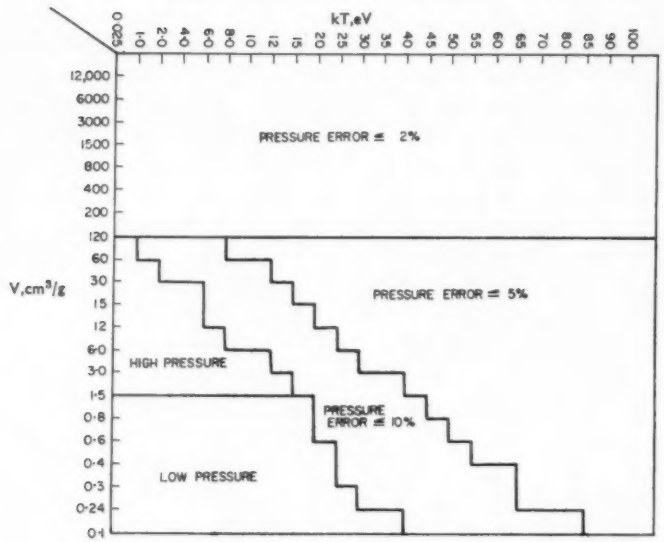


FIG. 4.—Schematic representation of pressure accuracy.

rapid, continuous rise in the number of free electrons from 25 per cent of all electrons at $T = 7$ to 72 per cent at $T = 8$ to 100 per cent at $T = 9$. At eight-fold liquid density, the last bound states of all species disappear together at $T = 5$ where the number of free electrons rises abruptly to 100 per cent from a negligible 0·75 per cent of all electrons at $T = 4{\cdot}5$. For higher densities there is complete ionization at all temperatures. These rapid changes are the coincidental results of the increase in particle density with temperature and the increase with particle density of the bound state electronic energies. The resulting abrupt composition changes resemble phase transitions, but do not appear to have any real physical basis.

C. *Discussion and evaluation of results*

At all volumes and temperatures for which the composition and free energy were calculated, the entropy, internal energy, enthalpy, Gibbs' free energy and pressure were also obtained. The reliability of these results is better measured by the accuracy of the pressure than the free energy, since the volume derivative is more sensitive to the deficiencies of the model and does not contain the arbitrary constants of the free energy. Our estimate of the accuracy of the calculated pressure is summarized schematically in Fig. 4. This estimate was made by a qualitative theoretical analysis of the limitations of the model and by comparisons with both experimental data and other theoretical calculations. Qualitative information concerning the limitations of the calculation was obtained by comparing our approximate model with the more exact considerations provided by quantum statistical theory. This information was used together with the numerical results to define regions of reliability of the model. For example, comparison with exact theory might indicate that Coulomb interactions are poorly represented by our model. The results which delineate regions of strong Coulomb interactions thereby allow us to define regions of questionable accuracy. Because of the lack of other independent data, it happens that for a large portion of $V - T$ space this process affords the only available means of evaluating the results obtained.

(1) *Limitations of the model.* In a rigorous derivation of the equilibrium properties of a hydrogen plasma, one considers a two-component system of electrons and protons which experience only Coulomb interactions. All possible combinations of simultaneous interactions among them would be evaluated. Because an exact solution of this problem is extremely difficult, much effort has been spent in developing approximate methods involving various expansions of the potential. In these expansions an attempt is made to find systematic groupings of the interactions between the particles based on the kinds of particle groups one expects to be important. One convenient classification of these groups is into bound or unbound systems. A group of particles is considered bound if there is a high probability that the distance between any two particles in the group is small compared to the radius of a sphere containing a mole of such groups. A group is unbound if any two particles in it have a high probability of being separated by a distance comparable to the radius of such a sphere. If we do not restrict the number of particles in these groups then

we get a complete description of the system, in which Coulomb interactions alone occur. However in this calculation we consider only the groups symbolized by H_2, H_2^+, H^-, H, H^+ and e^-, in the hope that these groups represent the most important bound-state systems. By including the electronic, vibrational, and rotational energies of these species, we are considering to some approximation the Coulomb interactions between the fundamental particles which resulted in the formation of these systems. In order to describe the unbound states of these groups and to account for the interactions of larger groups of fundamental particles, we have introduced several further approximations some of which involve non-Coulomb interactions. There are also approximations in our description of the bound states of H_2, and H_2^+. We assume the separability of electronic, vibrational and rotational energies, we use the cut-off harmonic oscillator and rigid rotor models for molecular vibration and rotation and we use only approximate electronic energies for the excited states of H_2. These approximations are most significant in regions VIII and IX of Fig. 1 where there are mostly atoms and molecules present.

Another approximation in the calculation is the use of the Debye-Hückel model for the collective Coulomb interactions of the four charged species assumed present. As is well-known, this approximation is accurate only in the limit of low charge density and high temperature. Moreover, when both the electron degeneracy and the Coulomb interactions are significant, there are additional errors due to the breakdown of the classical separation of kinetic and potential energy, and the neglect of quantum effects due to the wave nature of the electrons and the Pauli exclusion principle. No attempt has been made to include diffraction and exchange effects. However, we make a crude correction for the dependence of the potential energy of the electrons on their kinetic energy through our interpretation of the cut-off radius in the Debye expression. Since the Debye result with zero ionic radius is recovered in the limit of infinite charge density, the Coulomb interaction energy is given accurately only when this energy and the electron degeneracy are both small as in regions II, III and IV of Fig. 1. It should be progressively worse, the greater the charged particle density at a given temperature and the lower the temperature at a given density. i.e. error in regions V < VI < VII in Fig. 1.

In any model which, like the present one, includes only a few of the possible bound state systems, it is necessary to approximate in some way the Coulomb interactions of the remaining groups of particles. Part of this is accomplished by the use of the Debye model for the simultaneous interaction of all the charged groups present. To approximate the remaining interactions among the sub-groups to form larger groups, one common method is to assume some form for the potential of interaction which is a function of the configuration of the sub-groups considered as point particles. This approach is inexact since the energy of the entire group is not uniquely defined by the configurations of the centres of mass of the individual sub-groups. However this should lead to a reasonable description of the system in density temperature regions where the dimensions of the individual bound state systems are small compared to their centre of mass separations. This method then can describe only weakly attractive or repulsive forces, neglecting completely bound states of the larger groups. The present calculation approximates the interactions between the small sub-groups by a density dependent spherical confinment of the bound electrons of each group, which leads to a perturbation of their energy eigenvalues. We have then nothing on our model which corresponds to weak attractive forces, such as Van der Waals forces. In addition we overestimate the repulsive interaction by the extreme confinement of the bound electrons in a hard wall sphere whose volume is equal to the volume per group. However the model does give a qualitatively correct description of the effect of density on the electron distribution. As the individual sub-groups approach one another, then interact in such a way as to cause a redistribution of electrons resulting in delocalization. The reason for this is that a forced confinement of the electrons increases their kinetic energy more rapidly than their potential energy. Thus the minimum energy configuration subject to the density constraint must be associated with the maximum possible delocalization. This delocalization is synonomous with the loss of identity of the small sub-groups. In the present model, we have allowed for the disappearance of the sub-groups by a gradual decrease in the number of bound states. A bound state is no longer counted when its energy has increased to zero. This criterion is roughly equivalent to the exclusion of a state when its average volume exceeds the volume per group. While this gives us a qualitatively correct description of the density effect on the number of free electrons, our quantitative results are likely to have significant error due to our crude approximation of the energy-volume dependence. The approximation should be most reasonable when the perturbation due to the surrounding groups is small. Thus it should be valid over much of region VIII and IX of Fig.1 but becomes poorer as the density increases.

(2) *Comparison and discussion of results.*

(a) For a gas at normal volume down to a tenth of liquid density, it is believed that the calculated pressure is good to within a few per cent in the entire temperature range studied. This estimate is based on the following considerations:

(i) There is agreement with experimental heat capacities and pressures from the $T = 0{\cdot}025$ points at these volumes (*International Critical Tables*, 1928). Since only H_2 molecules are present at this temperature, this indicates that the model provides a reasonable description of the rotational and vibrational degrees of freedom. As a further test of the treatment of vibration, the harmonic oscillator eigenvalues were replaced by those obtained from a Morse potential. This change had almost no effect on the pressure and heat capacity, indicating that the detailed description of the vibrational states is not a significant factor in determining the results. This result, plus the fact that a previous calculation (HARRIS, 1959) with no cut-off gives very different answers shows that the counting of the vibrational states is much more important than their exact eigenvalues. Also, since we find that H_2 dissociates mainly in the ground electronic state, errors in our estimates of the excited states of H_2 can not lead to significant error.

(ii) The electrostatic interactions never exceed 5 per cent and the system is always classical in this region so that the Debye approximation is likely to be fairly accurate.

It seems that at present this method gives the only reasonable description of a multicomponent system in this region. The Thomas-Fermi statistical model (MARCH, 1957; METROPOLIS and REITZ, 1951; FEYNMAN, METROPOLIS and TELLER, 1949; LATTER 1955a, 1955b, 1956) is inapplicable for a dilute gas and the more rigorous diagram expansion methods have not made much progress as yet in regions where two, three and four-particle bound states make the most important contributions to the properties of the system (MAYER, 1937; MONTROLL, MAYER, 1941; MEERON, 1947 and 1958; FRIEDMAN, 1959a and 1959b; LEE and YANG, 1959; MONTROLL and WARD, 1958; GLASSGOLD, HECKROTTE and WATSON, 1959; LEVINE to be published; DEWITT, to be published). At temperatures where the system is completely ionized, so that existing diagram techniques can be applied, it is also a perfect classical gas and hence of little interest.

(b) As the density is increased at $T = 0{\cdot}025$, from a fifth of liquid density to twice liquid density the calculated pressures exceed the experimental values by an amount which increases rapidly from 30 per cent of the experimental pressure to a factor of 25. Only H_2 is present under these conditions, which means that F_3 and F_4 are zero. Since F_1 is exact, the error in the pressure is due entirely to $(\partial F_2/\partial v)$, i.e. to the assumed relation between F_2 and v.

It will be recalled that the energy of a hydrogen like orbital in a spherical box is almost unperturbed by a change in the radius of the box, down to a small radius. The energy then increases rapidly over a very short interval, producing a similar increase in the magnitude of F_2. Since the population of excited states is always small, this contribution to F_2 is due almost entirely to the volume dependence of the energies of the 1*s* orbitals. Furthermore the contribution to F_2 from the ground states of H_2, H_2^+, H^-, H is proportional to the number of bound groups, being otherwise independent of temperature. Therefore, since the number of bound groups decreases with increasing temperature and other contributions to the pressure increase both the relative and absolute magnitude of the error in the pressure caused by this aspect of the model goes down as the temperature rises.

At these densities and $T = 0{\cdot}025$, the experimental evidence indicates that the interactions of bound groups strongly affect the pressure. However, in the same way that the effect of F_2 on the calculated pressure becomes small, the relative contribution of other molecular interactions to the pressure must also decrease with temperature. Furthermore, when the system consists mainly of ions, the contribution of Coulomb interactions to F is found to be moderate and the system is classical. As a result of this analysis, a temperature region for this volume range i.e. $v = 3$–120 cm^3/g can be estimated where the error in pressure is less than 10 per cent and another where the error is further reduced to less than 5 per cent as indicated in Fig. 4.

Unfortunately, except for the low-temperature experimental data and the normal density Thomas-Fermi-Dirac calculation of COWAN and KIRKWOOD (MARCH, 1957; METROPOLIS and REITZ, 1951; FEYNMAN, METROPOLIS and TELLER, 1949; LATTER, 1955a, 1955b, 1956), there is nothing with which to compare our results in these regions. A comparison with the COWAN-KIRKWOOD results in Table 2 show that our pressures are too high at the lower temperatures. That the use of orbitals in a spherical box would lead to excessive pressures was obvious from the outset. While we were interested to discover quantitatively the error incurred, the model was used mainly to achieve pressure ionization in a simple way and for correct physical reasons.

(c) Four and eight-fold liquid density are special

TABLE 2.—COMPARISON OF COWAN-KIRKWOOD AND PRESENT CALCULATION H_2 AT NORMAL LIQUID DENSITY

PV/kT \ KT	100	70	50	30	20	10	8	5
CK	2·0	1·90	1·80	1·70	1·65	1·53	1·45	1·40
Present	2·0	1·98	1·96	1·90	1·88	1·96	1·94	1·9

volumes in this calculation since complete ionization occurs as the temperature increases due to the increase in particle density beyond a critical value. These are abrupt changes similar to phase transitions and give a discontinuity in the pressure. At temperatures lower than those required for ionization, the pressures are much too high, due again to the erroneously large dependence of the bound state energies on volume. After ionization, the contribution to the pressure from the Coulomb interactions increases drastically. This is the only region where the quantity a/R' become greater than one. The results at these volumes are not to be trusted until the potential energy has again become a moderate fraction of the kinetic energy. For $v = 3$ this means a minimum temperature of 10 eV and for $v = 1{\cdot}5$ a minimum temperature of 15 eV. The results of a Thomas-Fermi (MARCH, 1957; METROPOLIS and REITZ, 1951; FEYNMAN, METROPOLIS and TELLER, 1949; LATTER, 1955a, 1955b, 1956) calculation are better than ours at the lower temperatures at these volumes. Their pressures at zero temperature are still higher than experiment, but lower than ours. The *TF* results continue to be lower until 5 eV when ours become lower and stay lower even at 100 eV. This latter comparison indicates that our estimate of the interaction pressure is larger than that in the *TF* model.

(d) In the region of complete pressure ionization, $v \leqslant 0{\cdot}8$ cm³/g the results are very poor at low and moderate temperatures. This is a region of extreme degeneracy and large interactions where the neglect of quantum diffraction and exchange effects and the use of only the Debye term, altered with a cut-off which is not too effective for complete ionization severely limit the applicability of the model. Table 3 gives a comparison for $v \leqslant 0{\cdot}4$ and zero temperature of the results of GELL-MANN and BRUECKNER (1957), the present calculation and a *TF* calculation. If the more rigorous GELL-MANN, BRUECKNER results are to be taken as most accurate in the region of convergence of their series of terms, then it is apparent that our model severely overestimates the electrostatic interaction while the *TF* model underestimates it. Any improvement of our model in this region must come from a consideration of quantum effects on the

TABLE 3.—COMPARISON OF HIGH DENSITY ZERO TEMPERATURE RESULTS OF GELL-MANN-BRUECKNER, THOMAS-FERMI-DIRAC AND PRESENT CALCULATION

(cm³/g) \ Calculation	$0{\cdot}4(r_s^{**} = 1{\cdot}0)$			$0{\cdot}24(r_s = 0{\cdot}865)$			$0{\cdot}1(r_s = 0{\cdot}644)$		
	KE*	PE*	P†	KE	PE	P	KE	PE	P
Present	28·86	−28·18	1·47	40	−39·5	3·6	72·5	−70·5	17·6
GMB	28·53	−10·89	9·21	40	−15·8	25·8	72·5	−20·0	120
TFD			21·97			59·4			306

* Kinetic energy and potential energy in units of electron volts.
† Pressure in megabars.
** r_s = r_0/a where r_0 = volume per electron and a is the Bohr radius.

Coulomb interactions in the completely ionized plasma. In this region the usefullness of our model is extremely limited and the more rigorous calculations have made the most progress. At temperatures where the Coulomb interaction is again a small part of the kinetic energy our results should again become reasonable. In this high density-high temperature region our accuracy is comparable to that of the diagram calculations of DEWITT (MAYER, 1937; MONTROLL and MAYER, 1941; MEERON, 1958; FRIEDMAN, 1959a, 1959b; LEE and YANG, 1959; MONTROLL and WARD, 1958; GLASSGOLD, HECKROTTE and WATSON, 1959; LEVINE, to be published; DEWITT to be published) on the partially degenerate completely ionized plasma.

To sum up then it is felt that progress has been made in describing the equilibrium properties of a partially ionized, partially dissociated multicomponent plasma, in regions of V–T space where internal degrees of freedom and bound electronic states are important and in regions of moderate interaction and moderate degeneracy. Under conditions of complete pressure ionization, i.e. high density, low and moderate temperatures, this model is unsatisfactory. At higher temperatures where the kinetic energy terms again dominate, our model, the *TF* model and the diagram approach appear to be about equally accurate.

REFERENCES

FOWLER R. and GUGGENHEIM E. A. (1952) *Statistical Thermodynamics*, Cambridge University Press.

GELL-MANN M. and BRUECKNER K. A. (1957) *Phys. Rev.* **106**, 364.

GROOT S. DE and TEN SELDAM C. A. (1946) Physica, *s'Grav.* **12**, 669.

HARRIS G. M. (1959) *J. Chem. Phys.* **31**, 1211.

International Critical Tables, III (1928), 5.

MCDOUGALL J. and STONER E. C. (1938) *Trans. Roy. Soc. (Lond.)* **A237**, 67.

MARCH N. H. (1957) *Advances in Physics* **6**, No. 21, 1; METROPOLIS N. and REITZ J. R. (1951) *J. Chem. Phys.* **19**, 555; FEYNMAN R. P., METROPOLIS N. and TELLER E. (1949) *Phys. Rev.* **75**, 1561; LATTER R. (1955) *Phys. Rev.* **99**, 510; (1955) **99**, 1854; (1965) *J. Chem. Phys.* **24**, 280.

MAYER J. E. (1937) *J. Chem. Phys.* **5**, 67; MONTROLL E. W. and

MAYER J. E. (1941) *J. Chem. Phys.* **18,** 626; MEERON E. (1947) *J. Chem. Phys.* **26,** 804; (1958) **28,** 630; FRIEDMAN H. L. (1959) *Molecular Phys.* **2,** No. 1, 23; (1959) **2,** No. 2, 190; LEE T. D. and YANG C. N. (1959) *Phys. Rev.* **113,** 1165; MONTROLL E. W. and WARD J. C. (1958) *Phys. Fluids* **1,** 55; GLASSGOLD A. E., HECKROTTE W. and WATSON K. M. (1959) *Phys. Rev.* **115,** 1374; LEVINE H. *Phys. Fluids.* To be published; WITT H. E. DE. To be published.

APPENDIX 1

The Laguerre Polynomials provide energies for each hydrogenic state at a discrete set of radii of the confining spherical box. These radii correspond to integral values of the principal quantum numer, n. For the lowest energy states no energies can be obtained for finite radii greater than a few Ångströms. Nevertheless, exept for the ground states of each species, we have used only the discrete data afforded by the Laguerre polynomials together with the energy value at $R = \infty$, to determine a smooth variation of the energies with R:

$$E_{ki}(R) = 13{\cdot}595Z^2_{ki}/n_i^2 \,.\, [1 - \exp a(R)_{0i}/z_{ki} \times R - Z_{ki}R/R_{0i})]$$

where E_{ki} = energy of the ith bound state of the kth species in spherical box of radius R

z_{ki} = effective nuclear charge of the ith state of the kth species

R_{0i} = the radius at which the energy of ith state of the H atom becomes zero

a_i = a fitting factor for the ith state. of H atom

n_i = principal quantum number of the ith state of the H atom.

For the ground states, zeros of the more general hypergeometric functions were used (DE GROOT and TEN SELDAM, 1946) to determine more accurately the energy-volume relationship for large radii. This leads to a different dependence of E on R:

$$E_{k(1s)} = 13{\cdot}595Z^2_{k(1s)}[1 - X^{-(a+bX)}]$$

where a and b are parametric constants, here 2 and 1·116 respectively and $X = Z_{ki}R/R_{0i}$. This more exact expression for the energy of an H atom in a box then, gives a smaller perturbation for large R, and a larger value of $(\partial E/\partial R)$ for box sizes near the region of applicability of the Laguerre polynomials. The more exact relationship was applied only to the ground states since it was thought that before excited states became appreciably populated atoms and molecules would no longer be a significant part of the composition. This is certainly a good approximation for the molecular states and is least true for the states of the hydrogen atom.

VOL 2 196

J. Nucl. Energy, Part C: Plasma Physics, 1961, Vol. 2, pp. 235 to 238. Pergamon Press Ltd. Printed in Northern Ireland

IV. SUMMARY TALK OF IDEAS PRESENTED AT THE 1959 INTERNATIONAL PLASMA PHYSICS INSTITUTE

M. N. ROSENBLUTH
John Jay Hopkins Laboratory for Pure and Applied Science, General Atomic Division,
of General Dynamics Corporation, San Diego, California

FIRST of all I would like to thank our hosts from the University of Washington and from the Boeing Airplane Company. In particular, Dr. JAMES DRUMMOND has laboured indefatigably for many months to bring about this conference. I think we have all been impressed by the programme he has provided for us and by the general excellence of the papers presented at this meeting. In fact this very excellence provides me with a considerable problem since this field is so diverse and has been so well covered by the preceding speakers. I am faced by a sort of insoluble 'many-subject' problem which I can only treat in a very cursory fashion which may best be described as the Random Phrase Approximation.

One thing that has been quite apparent at this meeting is a dichotomy between two rather different subjects which I will describe by the somewhat inexact titles of 'quantum' and 'classical' plasma physics. In general those familiar with one of these fields are quite ignorant of the other. This is certainly true in my own case so I hope that my simple-minded comments on quantum plasmas will be forgiven. One may hope that the cross-fertilization provided by this meeting will prove fruitful.

I would now like to describe in a qualitative way some similarities and contrasts between the two disciplines. The basic similarity is, of course, that in both cases we are concerned with plasmas—that is to say a many-body system in which the particles interact through electric and magnetic forces. However the basic physical situations of the plasmas are quite different. Quantum plasmas are primarily studied in connexion with solid state physics while classical plasma studies are motivated by fusion reactor work, electron tube research and astrophysical applications. It is the conditions prevailing in these phsyical situations which of course determine the nature of the disciplines.

In the first place in the quantum plasma the electrons are degenerate which means that the exclusion principle is important and that spin and exchange effects must be considered. An equally important contrast is that in the classical plasma the potential energy between particles at an average interparticle separation is very small compared to the mean thermal kinetic energy. Indeed this must be so or in a classical problem the electrons would all merely fall into the ions! This means that only the electromagnetic fields of many particles moving together coherently can be of importance—hence the validity of equations stressing the collective behaviour of plasmas. Two equivalent approaches of this type are the Random Phase Approximation of BOHM and PINES and the Collisionless Boltzmann Equation.

On the other hand in metals the potential energy is somewhat greater than the Fermi kinetic energy. Actually they are quite comparable in magnitude. Nonetheless the basis of the calculations usually made, such as the high density expansion or the Random Phase Approximation, is again that the kinetic energy is large. It is paradoxical under the circumstances that such good results have been obtained.

Thus we may conclude that the basic physics of quantum plasmas is more complicated both due to quantum effects and to the fact that neither a weak nor a strong coupling approach is obviously justified. On the other hand while the basic equations of classical plasmas are simple the questions which we ask of them are very complex. Thus the study of quantum plasmas is primarily concerned with the properties of a true thermodynamic equilibrium and with small excitations around such an equilibrium. Because of the complex physics involved the answers to such problems are not simple. On the other hand the true thermodynamic equilibrium of a classical plasma is most uninteresting—a Maxwell Distribution uniformly filling all space (this is true irrespective of the presence of a static magnetic field).

Thus the heart of the classical plasma problem is that it is basically a non-equilibrium problem. For instance the inhomogeneity of the plasma is usually of importance. At this point it appears useful to coin a phrase to describe situations which are often of interest—*meta-equilibrium*. We have said that the only thermodynamic equilibrium is a uniform Maxwellian distribution. On the other hand we have also pointed out that since the interparticle interaction is usually rather small it follows that the collision frequency is very low compared to the collective frequencies of the plasma. However it is only these close

collisions which drive the system to thermodynamic equilibrium. Hence we may define a meta-equilibrium as any time-independent solution of the collisionless Boltzmann equation and Maxwell's equations. The effect of close collisions is then to make an adiabatic transition between different meta-equilibrium states. It is the existence, nature and stability of these meta-equilibria with which we are concerned. Since these are not true equilibria we are not able to use many of the powerful methods of thermodynamics and statistical mechanics.

This difference in the problems to be studied is clearly seen in the nature of the experimental information available. For the quantum plasma physicist the experimental facts are by and large well-known. His plasmas are contained in nice stable pieces of metal. On the other hand the classical plasmas which exist in nature are ordinarily turbulent and have resulted from very complicated processes. In fact one can easily prove by the virial theorem that any classical plasma which is not confined by rigid material walls externally imposed magnetic fields, or gravitation, must explode. A relatively simple meta-equilibrium is however provided by the natural radiation belts surrounding the earth. The experimentalist who wishes to create a laboratory plasma is also beset by many problems since as we shall see it is only with very carefully tailored meta-equilibria that he may hope for stability.

Moreover fantastic purity is required in order that cooling by impurities be tolerable. Thus a monomolecular layer of wall material will destroy a fusion plasma. The result of these difficulties is that the experimental information is still quite fragmentary and often deals with situations which are far from the idealized ones envisaged by the theorist.

To sum up it would appear that the principal task of quantum plasma theory is the development of an intermediate coupling approach which can make a quantitative check with the experimental observations. In the classical case on the other hand the weak coupling theory is valid and the problem may be stated as the development of the statistical mechanics of a system in which only collective modes are important. In particular the question arises to what extent these modes, without collisions, are capable of producing a true thermodynamic equilibrium.

I would now like to describe in somewhat more detail the present state of development of the theory of classical plasmas. For the sake of definiteness I shall tie the discussion to the consideration of quiescent inhomogeneous plasmas confined by a static magnetic field as in proposed fusion reactors. First a brief discussion of the justification for the collisionless Boltzmann equation. This can be written

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0 \qquad (1)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad (2)$$

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\ddot{\mathbf{E}} = -4\pi e \int \frac{\partial f}{\partial t}\mathbf{v}\, d^3\mathbf{v} \qquad (3)$$

This set of equations is of course exact if given the proper interpretation, namely that the distribution function f consists of a set of δ-functions at the points in phase space representing the positions of the particles. Then of course it is not very useful. Let us now conceptually smear out the δ-functions over a region in phase space large compared to the interparticle distance but small compared to regions of macroscopic interest. Then the same equations are applicable but f is now a continuous distribution function; e.g., $e^{-mv^2/2kT}$. This is the usual collisionless Boltzmann equation. Mathematically we may accomplish the approach to a continuous fluid by letting n, the particle density, become infinite in such a way that ne, nm and v remain constant. The exact equations may then be expanded in powers of $1/n$. The dimensionless expansion parameter turns out to be the inverse of the number of particles in a sphere of radius the Debye length $(mv^2/4\pi ne^2)^{1/2}$. For typical situations of interest this parameter may be 10^{-6}. To lowest order one obtains the collisionless Boltzmann equation and to next order the Fokker-Planck equation. Time scales for the collisional processes described by the Fokker-Planck equation are then 10^6 times as long as the collective processes described by the collisionless Boltzmann equation.

Let us now turn to the question of the existence of static meta-equilibrium solutions of equations (1)–(3). The operator in equation (1) may be interpreted as the total time derivative of f moving through phase space along the particle orbit which is governed by **E** and **B**. Hence if closed orbits exist in phase space a time-independent solution of (1) is simply that f be constant along such an orbit. In the case of magnetic fields possessing sufficient symmetry the existence of closed orbits can be shown and solutions of the above equations explicitly given.

For physically realizable fields with imperfect symmetry the orbits do not exactly close and an exact confined meta-equilibrium does not exist. However some beautiful work by KRUSKAL and GARDNER has shown that as long as the changes in the field are small within a particle radius of gyration then the orbits

come extremely close to being closed. This has been experimentally verified by LAUER and co-workers at the Lawrence Radiation Laboratory who have shown that the decay positrons from radioactive neon remain confined within a mirror machine for many millions of traversals. Thus it may be considered to have been proven that confined static meta-equilibria exist for all practical purposes.

Next we must study the stability of these equilibria. To date almost all work in this direction has been based on a linearized theory. That is to say we consider small perturbations around a static meta-equilibrium which is described by a distribution function f_0 and fields $\mathbf{E}_0$ and $\mathbf{B}_0$. If we denote the perturbations by subscript 1, then the linearized form of equation (1) becomes

$$\frac{df_1}{dt} = -\frac{e}{m}(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) \cdot \nabla_v f_0 \qquad (4)$$

where the time derivative is to be taken along the particle orbit in the unperturbed field. By performing the time integral and substituting in (3) an integro-differential equation is obtained for the perturbed field $\mathbf{E}_1$.

In general this equation is difficult to solve because of the complicated nature of the orbits in an arbitrary field. In two special cases the orbits are simple and the analysis has been performed. The first case, to which I will return later, is that of an infinite homogenous plasma in a constant magnetic field. Here the orbits are simple spirals and the integro-differential equation reduces to an algebraic dispersion relation for the frequency of the normal modes. The second simple case is obtained by restricting our attention to disturbances of low frequency and long wavelength such that the fields which a particle sees vary only slightly during a Larmor period. In this limit the particles remain 'frozen' on magnetic field lines and there exist adiabatic invariants of the motion, i.e. the magnetic moment and longitudinal action integral, such that the equations assume a simple character. In fact in this case the equations of fluid magneto-hydrodynamics, such as would apply to a liquid metal, remain valid except for the adiabatic law of pressure change. This becomes more complicated because of the existence of unimpeded heat flow along the lines of force but one is still able to construct a variational principle to test for stability based on the concept of virtual displacement of the field lines. One is led to the interesting result that a plasma is more stable with respect to these hydrodynamic-type modes than a liquid metal would be. Nonetheless the great majority of plasma containment devices are unstable to such perturbations. However certain stable configurations have been found e.g. the helically-wound stellarator, the inverted pinch, the cusp and the mirror machine with conducting end-plates.

Let us now return to the consideration of an infinite homogeneous plasma where the dispersion relations may be derived exactly. These are very rich dispersion relations and much has been learned from them. In the first place one can show easily that if the distribution function is isotropic and a decreasing function of energy, i.e. if f is only a function of v^2 and if $\partial f/\partial v^2 < 0$ then all the modes of excitation are stable. Hence we know that collective oscillations by themselves cannot play the role of collisions in producing a Maxwell Distribution.

A great many modes exist. In addition to the hydromagnetic modes we have already discussed there are electrostatic plasma oscillations, cyclotron waves, transverse electromagnetic waves and a variety of others. We will refer to these as the high-frequency modes.

Let us now look at the case where the distribution function is not isotropic. For example there may be an asymmetry in the components of the pressure tensor perpendicular and parallel to the magnetic field or there may be beams of particles moving along the field lines. Such beams might be produced by 'runaway' electrons which have been accelerated by a large electric field.

Where an anisotropy exists the high-frequency modes often become overstable, that is to say subject to growing oscillations. The mechanism for this is rather interesting. It depends on a trapping of individual particles by waves. Thus when the phase velocity of the wave equals the velocity of a particle, the latter moves along in phase with the wave and becomes trapped by it. If it was initially going faster than the wave it will give up energy in the course of the trapping and thereby feed a growing oscillation. If on the other hand there are more slow particles then the wave will be damped. Hence if a beam of particles is present the plasma is likely to be unstable with respect to waves moving slightly slower than the beam. Fortunately in the limit that the anisotropy becomes small the instabilities either vanish or their rate of growth becomes negligibly small.

The existence of these instabilities poses great problems in obtaining a hot plasma. On the one hand the distribution function must always be nearly isotropic in order to maintain stability. On the other hand the processes of breaking down the gas and subsequently confining and heating the plasma are likely to be rather violent, involving large accelerating electric

fields, and this violence may well lead to anisotropies. Hence a principal problem for the experimentalist is to learn the art of gentle handling of plasmas.

There remains one gap in our understanding of the linear stability of confined plasmas. That is the behaviour of the high-frequency modes in an inhomogeneous plasma. I would now like to describe some preliminary conclusions of recent work by NICK KRALL and myself. Our study of the homogeneous case has led to an understanding that except for the effects of the trapping process described above the plasma may be thought of as a substance with a real dielectric constant for which the square of the frequency remains real. Hence the introduction of a small inhomogeneity in the plasma will only lead to a frequency shift rather than overstability except for the effects of trapping.

However there is a new possibility for trapping which exists in an inhomogeneous plasma—namely, the particle drifts and currents perperdicular to the magnetic field. Such currents are of course essential to a confined plasma since it is the $\mathbf{j} \times \mathbf{B}$ force which supports the plasma pressure gradient. Thus we are led to expect that waves with a phase velocity comparable to the plasma drift velocity may become unstable.

Now the plasma drift velocities are much smaller than their thermal velocities. In fact if we describe the inhomogeneity as being of magnitude ϵ, then the drift velocities are equal to ϵ times the thermal velocities. Roughly speaking ϵ is the fractional charge of the magnitude of the field in a particle radius of gyration. The condition for trapping is then that the phase velocity of the wave equal the drift velocity

$$v_{\text{phase}} = \omega\lambda = \epsilon v_{\text{thermal}}. \tag{5}$$

Here λ is the wavelength of the disturbance and ω is a characteristic frequency of the plasma such as the Larmor frequency or the plasma frequency. It follows then that the possible unstable high frequency modes will have a very short wavelength for weakly inhomogeneous plasmas, proportional to ϵ. A more detailed analysis seems to support the above qualitative arguments and thus leads to the conclusion that even a hydrodynamically stable confined plasma with no pronounced anisotropies may be subject to very short wavelength instabilities.

We now come to the broadest area of ignorance in the entire field—namely the question of non-linear effects. While we may be able to ascertain that an equilibrium is subject to unstable small amplitude oscillations we have very little idea what the ultimate effect of the instabilities will be. Presumably they will always act in a direction to smooth out the inhomogeneities or anisotropies which are driving them. They are likely also to induce a kind of turbulence so that the plasma is far from the sort of static meta-equilibrium we have been discussing. This question is of particular interest for the short wave length instabilities which are characteristic of the high-frequency modes, since non-linear effects must predominate when the amplitude of the perturbation exceeds the wave length.

Some numerical work on a simple model by DAWSON and BUNEMAN suggests that what may happen is that once the non-linear stage has been reached then the wave quickly breaks and the energy is distributed to even shorter wavelengths eventually approaching some sort of thermalization among the particles. If this picture is correct then the plasma may be thought of as undergoing very localized boiling. A very crude estimate of the effects of such boiling can be made by constructing a diffusion coefficient equal to the square of the wavelengths of the instability times its linear growth rate. Because of the short wavelengths involved such diffusion coefficients are usually very small. On the other hand STURROCK has shown that there is also some tendency, at least in the early non-linear phases, for the energy to move towards longer as well as shorter wavelengths. If a sizable amount of energy can be built up in long wavelength oscillations then a more rapid diffusion would of course result. In addition to diffusion the possibility that such turbulence might lead to sizable coherent radiation being emitted from the plasma must also be considered. In view of the failure over many decades to produce a satisfactory *a priori* theory of ordinary hydrodynamic turbulence it seems clear that the answer to these questions must be provided in large part by experiment.

To sum up I think anyone would have to be impressed with the vast amount of work and new information which has come forth in the past few years concerning plasmas as well as other many-body systems. In the field of quantum plasmas we now have a good theoretical understanding of the basic equations which govern metallic plasmas. Even such an exotic effect as superconductivity is explained which appears to be a really remarkable triumph of the theory. The theory of high-temperature classical plasmas, at least in the neighbourhood of quiescent meta-equilibrium states, has been well formulated. And while no one is yet close to fusion power nonetheless kilovolt plasmas have been produced which persist for times several orders of magnitude longer than the time scales for the simple collective instabilities which beset the earliest work in this field.

VOL. 2 196

OL.
2
961